BOOLEAN EQUATIONS WITH MANY UNKNOWNS

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The method of constructing general analytical solutions of Boolean equations previously developed for equations with a single unknown is generalized to the case of many unknowns. The efficiency of the method is demonstrated using several examples. In particular, we establish the integrability conditions and the general form of the solution of a “Boolean partial differential equation”; we consider the design of an RS-flipflop circuit and determine the general form of the corresponding family of functions $R$ and $S$.

1. Introduction

Boolean equations often arise in contexts that involve mapping in spaces of binary vectors $E_2^k$, $E_2 = \{0, 1\}$. This category includes the equations of Boolean algebra; relationships describing binary relations having certain properties (transitivity, connectedness, asymmetry, etc.); Boolean inequalities; problems in the design of optimal switching circuits (e.g., circuits realizing some desired behavior using flipflops of a specified type), and many others.

Formally, the problem of solving Boolean equations is stated as follows. Given are $k$ Boolean relationships (a system of equations) of the form

$$f_i(x, y) = 0 \quad (i = 1, \ldots, k),$$

where $x = (x_1, \ldots, x_m)$ and $y = (y_1, \ldots, y_n)$ are 0-1 vectors in the spaces $E_2^m$ and $E_2^n$, respectively, $f_i$ are known Boolean functions

$$f_i : E_2^{m+n} \to E_2.$$

It is required to find a family of Boolean functions $\{\psi_i(x)\}_{i=1}^n$ and a domain $D \subset E_2^n$ such that substitution of these functions for the variables $y_i$ reduces the values of $f_i(x, \psi_1(x), \ldots, \psi_n(x))$ identically to zero on the set $D$, i.e., we have $f_i(x, \psi(x)) = 0 \quad \forall x \in D$, where following our convention we have replaced the system of functions $\psi_1(x), \ldots, \psi_n(x)$ with a single vector-function $\psi(x)$.

If we have found at least one function family $\{\psi_i(x)\}_{i=1}^n$, then we say that the system of equations (1) is solvable for $y$ on the set $D$. A distinctive feature of Boolean algebra is that any system of equations of the form (1) is representable by a single equation using the disjunctive sum of all the equations entering the original system. Specifically, the equation

$$\bigvee_{i=1}^n f_i(x, y) = 0$$

is equivalent to system (1). The system of Boolean inequalities

$$f_i(x, y) \leq g_i(x, y) \quad (i = 1, \ldots, k),$$

where $f_i : E_2^{m+n} \to E_2$, $g_i : E_2^{m+n} \to E_2$, is also reducible to an equation of this type, because each inequality in (4) is equivalent to the equation $f_i(x, y)g_i(x, y) = 0$, and the entire system reduces to a single relationship

\[ \sqrt[k]{\prod_{i=1}^{k} f_i(x, y) \bar{g}_i(x, y)} = 0, \] which differs from (3) only by the form of the defining functions. In what follows, we therefore solve a single equation

\[ f(x, y) = 0, \tag{5} \]

in which \( f : E_2^{m+n} \to E_2 \), and \( y = (y_1, \ldots, y_n) \) is the vector of unknowns for which analytical (logical) expressions have to be found.

The general solution of Eq. (5) for \( n = 1 \) is considered in [1]. The case of many variables \( (n > 1) \) has been studied in [2] and is traceable to the work of Akers [3], where solutions are obtained by the technique of "Boolean differential calculus." However, the analytical representation of solutions (by Boolean forms) is found in [2] with the aid of a tabular method (Karnaugh maps of functions), which restricts the applicability of the general solutions in cases with many variables. Contrary to the approach of [2], we describe procedures that find the general solution of Eq. (5) without resorting to "truth tables."

2. General Solution of Boolean Equation with Many Unknowns

Suppose that we have already solved Eq. (5) and let \( y_1(x), \ldots, y_n(x) \) be one of its solutions. Substitute the solutions \( y_2(x), \ldots, y_n(x) \) for the variables \( y_3, \ldots, y_r \) in \( f(x, y) \), and consider the equation

\[ f(x, y_1, y_2(x), \ldots, y_n(x)) = 0 \tag{6} \]

in one unknown \( y_1 \). This equation is solvable for \( y_1 \) if and only if [1]

\[ f_1(x, y_2(x), \ldots, y_n(x)) \overset{\text{def}}{=} f(x, 1, y_2(x), \ldots, y_n(x)) f(x, 0, y_2(x), \ldots, y_n(x)) = 0. \tag{7} \]

Its basis solutions are

\[ y_1^1(x) = f(x, 1, y_2(x), \ldots, y_n(x)), \tag{8} \]
\[ y_1^2(x) = f(x, 0, y_2(x), \ldots, y_n(x)), \]

and the general solution is

\[ y_1(x) = \alpha_1(x) y_1^1(x) \lor \bar{\alpha}_1(x) y_1^2(x), \tag{9} \]

where \( \alpha_1(x) \) is an arbitrary Boolean function of \( m \) variables.

Now consider (7) as an equation in \( y_2 \):

\[ f_1(x, y_2, y_3(x), \ldots, y_n(x)) = 0. \tag{10} \]

Similarly to the case of \( y_1 \), this equation is solvable for \( y_2 \) if and only if

\[ f_2(x, y_3(x), \ldots, y_n(x)) \overset{\text{def}}{=} f_1(x, 1, y_3(x), \ldots, y_n(x)) f_1(x, 0, y_3(x), \ldots, y_n(x)) = 0. \tag{11} \]