An Extremal Problem for Algebraic Polynomials With Zero Mean Value on an Interval

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Abstract. Let \( P_0^n(h) \) be the set of algebraic polynomials of degree \( n \) with real coefficients and with zero mean value (with weight \( h \)) on the interval \([-1, 1] \):
\[
\int_{-1}^{1} h(z)p_n(z) \, dz = 0;
\]
here \( h \) is a function which is summable, nonnegative, and nonzero on a set of positive measure on \([-1, 1] \). We study the problem of the least possible value
\[
i_n(h) = \inf\{\mu(p_n) : p_n \in P_0^n\}
\]
of the measure \( \mu(p_n) = \text{mes}\{z \in [-1, 1] : p_n(z) \geq 0\} \) of the set of points of the interval at which the polynomial \( p_n \in P_0^n \) is nonnegative. We find the exact value of \( i_n(h) \) under certain restrictions on the weight \( h \). In particular, the Jacobi weight
\[
h(\alpha, \beta)(z) = (1 - z)^\alpha(1 + z)^\beta
\]
satisfies these restrictions provided that \(-1 < \alpha, \beta \leq 0 \).

Key words: extremal problem, algebraic polynomials, polynomials with zero mean, Jacobi weight.

Introduction

Let \( P_n \) be the set of algebraic polynomials of degree \( n \) with real coefficients and let \( P_0^n = P_0^n(h) \) be the set of polynomials from \( P_n \) having zero mean with weight \( h \) on the interval \( I = [-1, 1] \):
\[
P_0^n(h) = \left\{ p_n \in P_n : \int_{-1}^{1} h(z)p_n(z) \, dz = 0 \right\};
\]
here \( h \) is a function summable and nonnegative on \( I \) and nonzero on a set of positive measure from \( I \). For the polynomial \( p_n \), we introduce the set \( E(p_n) = \{z \in I : p_n(z) \geq 0\} \) of points of the interval \( I = [-1, 1] \) at which the polynomial \( p_n \) is nonnegative and consider the Lebesgue measure of this set,
\[
\mu(p_n) = \text{mes} E(p_n) = \text{mes}\{z \in I : p_n(z) \geq 0\}.
\]
For any natural \( n \), we define the quantity
\[
i_n = i_n(h) = \inf\{\mu(p_n) : p_n \in P_0^n\}.
\]

In this paper we find the exact value of (2) under certain restrictions on the weight \( h \). A. G. Babenko studied problem (2) for \( h \equiv 1 \) and has shown \([1, \text{Theorem 1.2}]\) that
\[
\frac{1}{\sqrt{2n^2}} \leq i_n(1) \leq \frac{12}{n^2}.
\]
He obtained the estimate above by using the polynomial

$$R_{2k-1}(z) = \frac{P_k^2(z)}{z - x_k},$$  \hspace{1cm} (3)

where $P_k$ is the Legendre polynomial of degree $k$ and $x_k$ is its largest zero. In his paper [1, p. 28], Babenko writes that he borrowed the construction of the polynomial (3) from the paper by Bernshtein [2, p. 201] who noticed that the polynomial (3) has nonzero mean on $[-1, 1]$. In what follows, we show that the extremal polynomial of problem (2) has a similar form in the more general case as well.

Earlier Babenko [3] calculated the quantity

$$i_n^* = \min_{t_n \in T_n^0} \text{mes}\{x \in [0, 2\pi] : t_n(x) \geq 0\}$$  \hspace{1cm} (4)

which is the analog of $i_n(1)$ for the set $T_n^0$ of trigonometric polynomials $t_n$ of degree $n$ with zero mean on the period:

$$\int_0^{2\pi} t_n(x) dx = 0;$$

namely, he has shown that

$$i_n^* = \frac{2\pi}{n + 1}.$$

The last problem appeared in the paper by Taikov [4], where he studied the exact constant in the inequality between the norms of a trigonometric polynomial in the spaces $C[0, 2\pi]$ and $L(0, 2\pi)$. It is seen from Babenko's paper [5] that similar problems appear naturally in the proof of exact Jackson inequalities for the best mean square approximations of periodic functions by trigonometric polynomials. Problems of type (2) and (4) also arise in other fields of mathematics, in particular, in the derivation of the estimates of the density of set packings (see [6, 7] and references therein).

§1. Auxiliary statements

It is readily verified that the extremal polynomial of problem (2) on which the lower bound is reached exists. Indeed, the coefficients $a_1, \ldots, a_n$ of the polynomials

$$p_n(x) = \sum_{k=0}^{n} a_k x^k$$  \hspace{1cm} (5)

from the set $P_n^0$ can be chosen arbitrarily; after this, the coefficient $a_0$ is uniquely determined from the orthogonality condition

$$\int_{-1}^{1} h(x)p_n(x) dx = 0.$$

The function (1) is uniform with respect to the coefficients of $p_n$ (to be more exact, it is not changed by multiplication of the polynomial by a positive constant); therefore, in (2) we can restrict ourselves to the polynomials (5) from $P_n^0$ with the condition

$$\sum_{k=1}^{n} |a_k| = 1.$$  \hspace{1cm} (6)

Thus problem (2) is the problem of minimizing a continuous function of the coefficients $a_1, \ldots, a_n$ of the polynomial (5) on the compact set (6); therefore, the lower bound in (2) is reached, i.e. the extremal polynomial in (2) exists.