Estimates of the Number of Zeros of Some Functions With Algebraic Taylor Coefficients

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ABSTRACT. We prove two theorems about the number of zeros of analytic functions from certain classes that include the Siegel E- and G-functions. By using these theorems, we arrive at a new proof of the Gel'fond-Schneider theorem and improve the result that the numerical determinant does not vanish in the proof of the Shidlovskii theorem.

Key words: number of zeros of an analytic function, Gel'fond–Schneider theorem, Hilbert's seventh problem.

Let \( K \) be an algebraic number field of degree \( \kappa \), let \( \mathbb{Z}_K \) be the ring of integer numbers in \( K \), and let \( \text{Norm}(\alpha) \) be the norm of a number \( \alpha \in K \). We denote the multiplicity of the zero of an analytic function \( f(z) \) at a point \( z = \alpha \) by \( \text{ord}_{z=\alpha} f(z) \).

Suppose that the function

\[
f(z) = \sum_{\nu=0}^{\infty} a_{\nu} \frac{z^\nu}{\nu!}
\]  

satisfies the following conditions:

1) all coefficients \( a_{\nu} \in K \);
2) these coefficients and the conjugate numbers

\[
a_{\nu}^{[k]}, \quad k = 1, \ldots, \kappa, \quad a_{\nu}^{[1]} = a_{\nu},
\]

from the field \( K \) satisfy the estimates

\[
|a_{\nu}| < \Lambda_1 C_1^\nu, \quad \prod_{k=2}^{\kappa} |a_{\nu}^{[k]}| < \Lambda_2 C_2^\nu, \quad \nu = 0, 1, \ldots ;
\]

3) there exists a sequence \( \{q_n\} \) of nonzero integer numbers in \( K \) such that

\[
q_\nu a_{\nu} \in \mathbb{Z}_K
\]

and

\[
|\text{Norm}(q_{\nu})| < \Lambda_3 C_3^\nu, \quad \nu = 0, 1, \ldots ,
\]

where \( \Lambda_1, \Lambda_2, \Lambda_3, C_1, C_2, \) and \( C_3 \) are positive numbers.

Theorem 1. Suppose that the function \( f(z) \) satisfies all the cited conditions. Moreover, let \( N(R) \) be the number of its zeros in the disk \( |z| \leq R \) (counted with their multiplicities); in particular, \( N(0) = \text{ord}_{z=0} f(z) \). Then for any positive \( R \), the following inequality holds:

\[
(N(R) - N(0)) \ln \left( 2 + \frac{N(0)}{C_1 R} \right) \leq N(0)(1 + \ln(C_1 C_2 C_3)) + 3C_1 R + \ln(\Lambda_1 \Lambda_2 \Lambda_3).
\]

In particular, Theorem 1 can be applied to the Siegel E-functions if they are understood in the sense of the definition in [1, Chap. 13, §1].

A similar theorem holds for another class of functions, which contains the Siegel \( G \)-functions (see the final remarks in the end of the monograph [1], as well as Siegel's classical paper [2]).

Let us choose a function
\[
g(z) = \sum_{\nu=0}^{\infty} a\nu z^{\nu}
\]
with coefficients \( a\nu \) satisfying the same conditions as the coefficients of (1).

The formal distinction between \( f(z) \) and \( g(z) \) is that the latter does not contain \( \nu! \). However, \( g(z) \), in contrast to \( f(z) \), is not an entire function.

**Theorem 2.** Suppose that \( g(z) \) satisfies the above-cited conditions and \( N(R) \) is the number of its zeros in the disk \( |z| \leq R \) (counted with their multiplicities); in particular, \( N(0) = \text{ord}_{z=0} g(z) \). Then for any \( R \) such that \( 0 < R < 1/(3C_1) \), the following inequality is satisfied:
\[
(N(R) - N(0)) \ln \frac{1}{3C_1 R} \leq N(0) \ln(1.5C_1 C_2 C_3) + \ln(3A_1 A_2 A_3).
\]

**Proof of Theorem 1.** Let \( p = N(0) \); then we have
\[
f(z) = \sum_{\nu=0}^{\infty} a\nu z^{\nu}, \quad a_p \neq 0,
\]
in (1), and hence,
\[
|\text{Norm}(q_p a_p)| \geq 1,
\]where the \( q_n \) are the numbers determined by (3). Hence,
\[
|a_p| \geq |\text{Norm}(q_p a_p)|^{1/2} \cdots |a[x]|^{-1} \geq (A_2 A_3)^{-1}(C_2 C_3)^{-p}.
\]

Now let \( \alpha_1, \ldots, \alpha_n \) be all the other zeros of \( f(z) \) lying in the disk \( |z| \leq R \), that is,
\[
|\alpha_j| \leq R, \quad f(\alpha_j) = 0, \quad \alpha_j \neq 0, \quad j = 1, \ldots, n, \quad n = N(R) - N(0).
\]
Then for any positive \( r \) and \( \lambda \), it follows from (7) that
\[
a_p = f^{(p)}(0) = \frac{p!}{2\pi i} \oint_{|z|=r} \frac{f(z)}{z^{p+1}} \, dz,
\]
\[
f(z) = \frac{1}{2\pi i} \oint_{|u|=\lambda p+3R} \left( \sum_{\nu=0}^{\infty} a\nu u^{\nu} \right) \prod_{j=1}^{n} \frac{u - \alpha_j}{u - \alpha_j} \, du = \frac{f(z)}{u - z}.
\]

By (1) and (2), we have
\[
|f(u)| \leq \sum_{\nu=0}^{\infty} A_1 C_1^\nu |u|^{\nu} \leq A_1 e^{C_1(\lambda p + 3R)} \quad \text{for} \quad |u| = \lambda p + 3R, \quad \prod_{j=1}^{n} \left| \frac{u - \alpha_j}{u - \alpha_j} \right| \leq \left( \frac{R + r}{\lambda p + 2R} \right)^n;
\]
therefore, (9) implies
\[
|a_p| \leq \frac{p!}{(2\pi)^2} 2\pi r \cdot \frac{1}{r} 2\pi (\lambda p + 3R)^{1-p} \left( \frac{R + r}{\lambda p + 2R} \right)^n A_1 e^{C_1(\lambda p + 3R)(\lambda p + 3R - r)^{-1}}.
\]

Since the number \( r > 0 \) is arbitrary, it follows that
\[
|a_p| \leq A_1 \lambda^{-p} \left( 2 + \frac{\lambda p}{R} \right)^{-n} e^{C_1(\lambda p + 3R)}.
\]

This inequality, in conjunction with (8) and the relations \( p = N(0) \) and \( n = N(R) - N(0) \), yields the estimate
\[
(N(R) - N(0)) \ln \left( 2 + \frac{\lambda N(0)}{R} \right) \leq N(0) (C_1 \lambda - \ln 3 + \ln(C_2 C_3)) + 3C_1 R + \ln(A_1 A_2 A_3).
\]

Setting \( \lambda = C_1^{-1} \) in (10), we obtain inequality (5). The proof is complete. \( \square \)

Here and in the proof of Theorem 2 the argument is closely related to that used in the proof of the Jensen theorem (see, e.g., [3, Chap. 1, §1.1]).