BRIEF COMMUNICATIONS

On a Class of Delay Functional-Differential Equations in Hilbert Space

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KEY WORDS: delay functional-differential equation, equations of neutral type, variable coefficients.

In the present paper we study functional-differential equations (FDE) of neutral type in a Hilbert space in the case of unbounded operator-valued variable coefficients and variable delays. For these equations, we establish the correctness of initial value problems in weighted Sobolev spaces in the half-line. Examples are presented showing that the assumptions of the main theorems are sharp.

Suppose that $\mathcal{H}$ is a separable Hilbert space, $A$ is a self-adjoint positive operator in $\mathcal{H}$ with bounded inverse, and $I$ is the unit operator in $\mathcal{H}$. We make the domain $\text{Dom}(A^\beta)$ of the operator $A^\beta$ ($\beta > 0$) a Hilbert space $\mathcal{H}_\beta$ by introducing the norm $\| \cdot \|_\beta = \| A^\beta \cdot \|$ on $\text{Dom}(A^\beta)$.

Let $L_{2,\gamma}((a, b), \mathcal{H})$ ($-\infty \leq a < b \leq +\infty$) be the space of vector functions ranging in $\mathcal{H}$, equipped with the norm

$$
\| f \|_{L_{2,\gamma}(a, b)} \equiv \left( \int_a^b \exp(-2\gamma t)\| f(t) \|^2 \, dt \right)^{1/2}, \quad \gamma > -\alpha_0,
$$

where $\alpha_0$ is the greatest lower bound [1] of the operator $A$ and $W^1_{2,\gamma}((a, b), A)$ is the weighted Sobolev space of vector functions such that $A^j u^{(1-j)}(t) \in L_{2,\gamma}((a, b), \mathcal{H})$ ($j = 0, 1$), equipped with the norm

$$
\| u \|_{W^1_{2,\gamma}(a, b)} \equiv \left( \| u^{(1)} \|^2_{L_{2,\gamma}(a, b)} + \| Au \|^2_{L_{2,\gamma}(a, b)} \right)^{1/2}.
$$

Here and in following we write

$$
v^{(j)}(t) \equiv \frac{d^j}{dt^j} v(t), \quad j = 1, 2, \ldots, \quad W^1_{2,0} \equiv W^1_2, \quad L_{2,0} \equiv L_2.
$$

For more details about the space $W^1_2((a, b), A)$ see [2, Chap. 1].

Consider the problem

$$
\Delta u \equiv \frac{du}{dt} + Au(t) + \sum_{j=1}^n \sum_{p=0}^1 (B_{jp}(t)S_{g_j}(A^{1-p}u^{(p)})(t)) = f(t),
$$

(1)

$$
u(+0) = \varphi_0 \quad (2)
$$
on the half-line $\mathbb{R}_+ = (0, +\infty)$. Here the $B_{jp}(t)$ ($j = 1, 2, \ldots, n$, $p = 0, 1$) are strongly continuous operator functions ranging in the ring of bounded operators in $\mathcal{H}$ and $\varphi_0 \in \mathcal{H}_{1/2}$. The operators $S_{g_j}$ are defined as follows:

$$(S_{g_j}v)(t) = \begin{cases} 
 v(g_j(t)) & \text{for } t \text{ such that } g_j(t) \geq 0, \\
 0 & \text{for } t \text{ such that } g_j(t) < 0,
\end{cases}
$$

where the $g_j(t)$ are real continuously differentiable functions such that $g_j(t) \leq t$ and $g_j^{(1)}(t) > 0$, $t \in [0, +\infty)$. By $g_j^{-1}(t)$ we denote the inverse of the function $g_j(t)$ and write $h_j(t) = t - g_j(t)$.


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Definition. Let $\gamma \geq 0$. A vector function $u(t) \in W^1_{2,\gamma}(\mathbb{R}_+, A)$ is called a **strong solution of problem** (1), (2) if $u(t)$ satisfies Eq. (1) almost everywhere on $\mathbb{R}_+$ and condition (2) holds in the sense of convergence in the space $\mathcal{F}_{1/2}$.

Let us introduce the following notation:

$$b_{jp}(\gamma) = \sup_{t \in [\varepsilon_j^{-1}(0), +\infty)} \left( \exp(-\gamma h_j(t)) \|B_{jp}(t)\|(g^{(1)}_j(t))^{-1/2} \right),$$

$$r_p(\gamma) = \sup_{\lambda : \mathbb{R} \lambda \geq \gamma} \|\lambda^p A^{-p}(\lambda I + A)^{-1}\|, \quad p = 0, 1; \quad j = 1, 2, \ldots, n; \quad \gamma > -\alpha_0.$$

**Theorem 1.** Suppose that there exists a $\gamma_0 \geq 0$ such that

$$\sigma(\gamma_0) < 1,$$

where

$$\sigma(\gamma) = \sum_{p=0}^{1} \sum_{j=1}^{n} r_p(\gamma) b_{jp}(\gamma).$$

Then for any $\gamma \geq \gamma_0$ the operator $V_\gamma$ defined by $V_\gamma u \equiv (2u, u(+0))$ maps $W^1_{2,\gamma}((a, b), A)$ onto $L^2_{2,\gamma}(\mathbb{R}_+, \mathfrak{F}) \oplus \mathfrak{F}_{1/2}$ and has a bounded inverse.

**Theorem 2.** Suppose that the inequality

$$\Delta \equiv \sum_{p=0}^{1} \sum_{j=1}^{n} \left( \lim_{t \to +\infty} \|B_{jp}(t)\|(g^{(1)}_j(t))^{-1/2} \right) < 1$$

is satisfied. Then the assertion of Theorem 1 holds with $\gamma_0 = 0$; moreover, if $\gamma_0 = 0$ and $f(t) \in L^2_{2}(\mathbb{R}_+, \mathfrak{F})$, then

$$\|u(t)\|_{1/2} \to 0, \quad t \to +\infty.$$

The following theorem pertains to the case of negative $\gamma$, that is, to the case of exponentially decreasing solutions.

**Theorem 3.** Suppose that $B_{j1}(t) \equiv 0$ $(j = 1, 2, \ldots, n)$ and $B_{j0}(t) = D_{jo}(t)C_j$, where the $C_j$ are compact operators in $\mathfrak{F}$ and the $D_{jo}(t)$ $(j = 1, 2, \ldots, n)$ are strongly continuous bounded operator functions in $\mathfrak{F}$. Assume also that inequality (4) is satisfied and there exist positive constants $\theta_p$ $(p = 0, 1)$ such that $\theta_0 \leq h_j(t) \leq \theta_1$ $(j = 1, 2, \ldots, n)$.

Then there exists a $\delta > 0$ such that for each $\gamma > \max(-\alpha_0, -\delta)$ the operator $V_\gamma u \equiv (2u, u(+0))$ maps $W^1_{2,\gamma}(\mathbb{R}_+, A)$ onto $L^2_{2,\gamma}(\mathbb{R}_+, \mathfrak{F}) \oplus \mathfrak{F}_{1/2}$ and has a bounded inverse.

Along with problem (1), (2), consider the following initial value problem:

$$\frac{du}{dt} + Au(t) + \sum_{p=0}^{1} \sum_{j=1}^{n} (B_{jp}(t)A^{-p}u^{(p)}(g_j(t))) = f_0(t), \quad t \in \mathbb{R}_+, \quad (5)$$

$$u^{(p)}(t) = y_p(t), \quad t \in \mathbb{R}_- = (-\infty, 0), \quad p = 0, 1, \quad u(+0) = \varphi_0. \quad (6)$$

Problem (5), (6) can be reduced to (e.g., see [3]) a problem of the form (1), (2) whose right-hand side possesses the representation

$$f(t) = f_0(t) - \sum_{p=0}^{1} \sum_{j=1}^{n} (B_{jp}(t)T^{g_j}(A^{-p}y_p)(t)), \quad (7)$$

where the operators $T^{g_j}$ are defined as follows:

$$(T^{g_j} v)(t) = \begin{cases} 0 & \text{for } t \text{ such that } g_j(t) \geq 0, \\ v(g_j(t)) & \text{for } t \text{ such that } g_j(t) < 0. \end{cases}$$