AN APPLICATION OF THE NEYMAN–PEARSON LEMMA TO
GAUSSIAN PROCESSES

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Let \( \xi_i(t), i = 1, 2, t \in [0, 1], \) be Gaussian zero mean processes with continuous sample paths. Bounds for the probabilities \( \beta_i = P\{\xi_i - a_i \in B\}, i = 1, 2, \) are given, where \( a_i \in C[0, 1] \) and \( B \) is a Borel subset of \( C[0, 1] \).

Bibliography: 5 titles.

Let \( \eta_i(t), i = 1, 2, t \in [0, 1], \) be Gaussian processes with zero means and continuous sample paths. Let \( a_i(t) \) be continuous functions (elements of the space \( C[0, 1] \) of all functions continuous on \([0, 1]\) provided with the standard uniform norm), and let \( B \) stand for a Borel subset of this space. In this paper, we give inequalities for the mutual closeness of the probabilities

\[
\beta_i = P\{\xi_i(t) - a_i(t) \in B\}, \quad i = 1, 2. \tag{1}
\]

For example, the probabilities for a sample path of the process \( \eta_i \) to be under a curve \( a_i(t) \) or inside a tube are just of this kind. The whole problem is in a natural way divided into two subcases:

(A) Correlation functions of the processes \( \eta_i \), \( i = 1, 2, \) coincide, \( a_i(t) \) are different; we consider \( \xi_1(t) \equiv \xi_2(t) \equiv \xi(t) \).

(B) Correlation functions of \( \eta_i \), \( i = 1, 2, \) differ, \( a_i(t) \equiv 0 \).

Case A. By definition, put \( \eta_i(t) = \xi(t) - a_i(t), a(t) = a_2(t) - a_1(t) \). For the sake of convenience we shall say that under the hypothesis \( H_0 \) the random process \( \eta(t), t \in [0, 1], \) has the same distribution as the process \( \xi(t) \), and under the alternative \( H_1 \) it has the same distribution as the process \( \eta_2(t) = \eta_1(t) + a(t) \). Let \( P_i, i = 1, 2, \) be the corresponding probability distributions in \( C[0, 1] \). Assume that these distributions are equivalent measures and hence

\[
dP_2/dP_1 = \exp\{\xi \Delta - \Delta^2/2\}.
\]

Here the random variable \( \xi \) has the standard Gaussian distribution \( N(0, 1) \) under \( H_0 \), and has the Gaussian distribution \( N(\Delta, 1) \) under the alternative \( H_1 \), \( \Delta = \Delta(a) > 0 \) is the value of a functional depending on the parameters of \( \eta_i(t) \). Let the test \( \psi \) reject \( H_0 \) if \( \eta(t) \not\in B \). Obviously, its significance is equal to \( 1 - \beta_1 \), and its power is equal to \( 1 - \beta_2 \).

According to the Neyman–Pearson lemma, for the most powerful test we have for some \( \Lambda \)

\[
1 - \beta_1 = P_1\left\{\log \frac{dP_2}{dP_1} \geq \Lambda\right\}, \quad 1 - \beta_2 \leq P_2\left\{\log \frac{dP_2}{dP_1} \geq \Lambda\right\},
\]

where

\[
\Lambda = \Phi^{-1}(\beta_1)\Delta - \Delta^2/2, \quad (1 - \beta_2) \leq 1 - \Phi(\Phi^{-1}(\beta_1) - \Delta)
\]

(here \( \Phi \) denotes the distribution function of the standard Gaussian law \( N(0, 1) \)). Taking into account the symmetry, we get the following relation:

\[
\sup_B |\Phi^{-1}(\beta_1) - \Phi^{-1}(\beta_2)| = \Delta, \tag{2}
\]
the upper bound being taken over all the Borel subsets of $C[0, 1]$. Obviously, this supremum is achieved if $\beta_1 = P\{x < \xi\}$ for some real $x$.

Let a measurable functional $F$ be defined on the space $C[0, 1]$. It is known that the boundedness of the value

$$
\sup_x p(x) < +\infty, \quad p(x) = \lim_{h \to 0^+} h^{-1} P\{F(\xi(t)) \in [x, x + h]\},
$$

permits one to obtain bounds of the convergence rate of distributions of this functional in the invariance principle. It also means the boundedness of the distribution density of the random variable $F(\xi(t))$. By $F$ denote the subset of all admissible shifts for the distribution in $C[0, 1]$ of the random process $\xi$. It is known that $F$ is a nondegenerate subspace, and $a \in F$. Assume that

$$
P\{F(\xi(t)) < x + h\} \leq P\{F(\xi(t)) - ha(t) < x\} + C(x, h) \cdot h, $$

$$
\sup_{x} \lim_{h \to 0^+} C(x, h) = C < +\infty.
$$

This assumption means that the derivative of $F$ in an admissible direction is separated from zero. Note that (3) implies (4) in the case of a Lipschitz functional $F$. Conversely, let (4) hold. Using (2), we get

$$
p(x) \leq C + \lim_{h \to 0^+} h^{-1}(\beta_2 - \beta_1) \leq C + \frac{\Delta(a)}{\sqrt{2\pi}} \exp\{-\frac{1}{2}|\Phi^{-1}(\beta_1)|^2\},
$$

where $\beta_1 = P\{F(\xi(t)) < x\}$, $\beta_2 = P\{F(\xi(t) - ha(t)) < x\}$.

Example 1. Let $K(t, s)$ be the correlation function of a Gaussian random process $\xi(t)$, and $F(\xi(t)) = \sup\{\xi(t)\} \xi(t)$. The correlation function $K(t, s)$ is continuous because of the sample continuity of $\xi(t)$. Assume that $K(t, s) \neq 0$ for any $t$. We divide the interval $[0, 1]$ into intervals $\delta_k$ such that for every $k$ we have $K(t, s) > 0$ on $\delta_k \times \delta_k$. For some positive $c$ we have $a_k(t) = c \int_{\delta_k} K(t, s) ds \geq 1$ on $\delta_k$. The function $a_k(t)$ is an admissible shift of the distribution of $\xi(t)$ on $\delta_k$ since $a_k(t) \in BL_2(\delta_k) \subset B^1_{\infty} L_2(\delta_k)$, the latter space being the set of all admissible shifts $[1]$. Here $B$ stands for the integral operator with the kernel $K(t, s)$.

We get further

$$
P\{\sup_{[0, 1]} \xi(t) \in [x, x + h]\} \leq \sum_k P\{\sup_{\delta_k} \xi(t) \in [x, x + h]\},
$$

$$
P\{\sup_{\delta_k} \xi(t) < x + h\} \leq P\{\sup_{\delta_k} (\xi(t) - ha_k(t)) < x\};
$$

hence, by the above, the distribution density of $\sup_{[0, 1]} \xi(t)$ is bounded.

Example 2. Let $F(\xi(t)) = \int_0^1 g(\xi(t)) dt$, $g(x)$ be a differentiable function, $g'(x) \geq 1$, and for some $\varphi \in L_2[0, 1]$, $\varphi \neq 0$,

$$
a(t) = \int_0^1 K(t, s) \varphi(s) ds \geq 0.
$$

Evidently,

$$
F(\xi(t)) - h \leq F(\xi(t) - ca(t)h), \quad c = \left(\int_0^1 a(t) dt\right)^{-1}.
$$

It follows that (4) holds, hence the distribution density of $F(\xi(t))$ is bounded.

Examples 1 and 2 are simple illustrations and have been considered in [3, 4] in more detail.

Lemma 1. The following inequality holds:

$$
\frac{1}{\sqrt{2\pi}} \exp\{-\frac{1}{2} |\Phi^{-1}(z)|^2\} \leq \sqrt{2\pi} \log \frac{1}{z}, \quad z = \min(x, 1 - x).
$$