ESTIMATION OF THE PARAMETERS OF A BIVARIATE DISTRIBUTION OF GAUSSIAN RANDOM VARIABLES THAT CANNOT BE OBSERVED SIMULTANEOUSLY

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A point estimator is proposed for certain numerical characteristics of a bivariate normal distribution with parameters that cannot be observed simultaneously.

In reliability theory, one frequently needs to estimate the joint distribution of several random parameters that cannot be measured simultaneously. For example, such parameters are the “expected lifetime” of a product under specific conditions, the electrical insulation properties of a material under specific operating conditions, etc. Plans for experiments allowing one to estimate some characteristics of the joint distributions were suggested in [1]. Then they were developed in the thesis [2] (see also [3-8]).

In the present paper, on the basis of one of these plans, we obtain a point estimate of certain numerical characteristics of a bivariate normal distribution of the parameters \( \xi_1 \) and \( \xi_2 \) of a product in the case these parameters cannot be observed simultaneously, where \( \xi_1 \) is the resistance of the product to momentary load, i.e., \( \xi_1 \) is the maximal load which does not collapse the product.

Let us consider the following experiment. Arrange the given products into pairs at random. Let \( (\xi_1^i, \xi_2^i) \), \( i = 1, 2 \), be the values of \( \xi_1, \xi_2 \) of an arbitrarily chosen pair, respectively. We measure the minimum value \( \eta_i = \min[\xi_1^i, \xi_2^i] \). As a result, one of the products collapses, and the other remains good. (We disregard the events \( [\xi_1^i = \xi_2^i] \) and consider \( \xi_i \) to be a continuous random variable.) Then we measure the value \( \xi_2 \) of the remaining product, i.e.,

\[
\eta_2 = \begin{cases} 
\xi_2^i, & \xi_1^i < \xi_2^i, \\
\xi_1^i, & \xi_1^i > \xi_2^i.
\end{cases}
\]

After the experiment, a realization of the random vector \((\eta_1, \eta_2)\) becomes known. Under the information given, we estimate the probability density

\[
h(x, y) = \frac{\partial^2}{\partial x \partial y} P\{\eta_1 < x, \eta_2 < y\}.
\]

It is not difficult to demonstrate that the densities \( h(x, y) \) and

\[
f(x, y) = \frac{\partial^2}{\partial x \partial y} P\{\xi_1 < x, \xi_2 < y\}.
\]

are related as follows:

\[
h(x, y) = 2f_1(x) \int_{-\infty}^{+\infty} f(\xi, y) \, d\xi,
\]

where

\[
f_1(x) = \frac{d}{dx} P\{\xi_i < x\}, \quad i = 1, 2.
\]

Let us apply relation (1) to the case where the random variables \( \xi_1, \xi_2 \) observed simultaneously are distributed according to the bivariate normal law

\[
f(x, y) = \frac{1}{2\pi\sigma_1\sigma_2(1-\rho^2)^{1/2}} \exp\left\{-\frac{X^2 - 2\rho XY + Y^2}{2(1-\rho^2)}\right\},
\]

where


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\[ X = \frac{z - m_1}{\sigma_1}, \quad Y = \frac{y - m_2}{\sigma_2}, \quad m_i = E\xi_i, \quad \sigma_i = D\xi_i, \quad i = 1, 2; \quad \rho = \text{cov} (\xi_1, \xi_2). \]

Consider the problem of testing hypothesis (2) and estimating the numerical characteristics appearing in (2) from the results of the experiments described above. Assume that \( n \) pairs of products have been tested and the data \((x_i, y_i), i = 1, \ldots, n\), have been obtained.

First, we find the distribution \( h(x, y) \) of the parameters observed that corresponds to law (2). We point out that we are solving an inverse problem. Actually, after experimenting one constructs the distribution \( h \) and then \( f \). Then it seems to be more natural to set up a hypothesis on \( h \) instead of \( f \). But this approach is unlikely to prove itself. The fact is that we cannot take any bivariate law as \( h \). For example, the distribution \( h \) cannot be normal of type (2), since, by virtue of (1), a function \( f \) taking negative values corresponds to it.

Let us prove first that

\[ J(b) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Phi(ax + b)e^{-x^2/2} \, dx = \Phi \left( \frac{b}{\sqrt{1 + a^2}} \right), \quad (3) \]

where

\[ \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\xi^2/2} \, d\xi. \]

It is easy to show that

\[ J(-\infty) = 0, \quad \frac{dJ(b)}{db} = \frac{1}{\sqrt{2\pi(1 + a^2)}} \exp \left( -\frac{b^2}{2(1 + a^2)} \right). \]

These two relations yield (3)

After substituting (2) into (1) we write

\[ h(x, y) = 2f_1(x)f_2(y) \int_x^\infty f \left( \frac{\xi}{y} \right) \, d\xi = \frac{1}{\pi\sigma_1\sigma_2} \Phi \left( \frac{\rho Y - X}{\sqrt{1 - \rho^2}} \right) \exp \left( -\frac{1}{2}(X^2 + Y^2) \right). \quad (4) \]

Thus, to verify whether the data agree with the hypothesis that (4) is equivalent to (2), we may apply the chi-square goodness-of-fit test. Using Eq. (3), we obtain the marginal distributions

\[ h_1(X) = \frac{d}{dX} P \{ \eta_1 < X \} = \frac{1}{\sigma_1} \sqrt{\frac{2}{\pi}} e^{-X^2/2} \Phi(-X), \]

\[ h_2(Y) = \frac{d}{dY} P \{ \eta_2 < Y \} = \frac{1}{\sigma_2} \sqrt{\frac{2}{\pi}} e^{-Y^2/2} \Phi \left( \frac{Y}{\sqrt{2 - \rho^2}} \right). \]

These laws belong to one class, and by rescaling they can be expressed in terms of the function

\[ h(X, a) = \sqrt{\frac{2}{\pi}} e^{-X^2/2} \Phi(aX), \quad a \in \mathbb{R}. \]

Note that for any value of the parameter \( a \) the function \( h(x, a) \) behaves as a probability density, and \( h(x, 0) = \sqrt{2\pi} \exp(-x^2/2) \). Since the distribution \( h(x, a) \) is an interesting generalization of the normal law, which is of great value in experiment design with Gaussian components that cannot be observed simultaneously, it is worthwhile to study it in greater detail.

**Table 1.**

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