A BOUNDARY-VALUE PROBLEM FOR PARABOLIC EQUATIONS WITH GENERAL NONLOCAL CONDITIONS

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We study a boundary-value problem with general two-point conditions with respect to the time coordinate, and periodic conditions on the spatial coordinates for Shilov-parabolic equations with constant coefficients. We construct the solution in the form of a Fourier series. We establish conditions for existence and uniqueness of a classical solution of the problem. We prove quantitative theorems on a lower bound for the small denominators that arise in solving the problem.

Problems of the type described in the title were have been studied [4, Ch. 5] for various partial differential equations, not including parabolic equations. The book just cited also contains a survey and bibliography of papers related to nonlocal problems for partial differential equations and operator-differential equations. Among more recent studies relating to this theme we note the papers [3, 5]. We shall use the following notation below; \( \Omega^p \) is the \( p \)-dimensional torus \( \{ x \in \mathbb{R}^p : 0 \leq x_r \leq 2\pi, r = 1, \ldots, p \} \), \( D = [0, T] \times \Omega^p \), \( k = (k_1, \ldots, k_p) \in \mathbb{Z}^p \), \( |k| = |k_1| + \cdots + |k_p| \), \( (k, x) = k_1 x_1 + \cdots + k_p x_p \), \( (k, k) = |k|^2 \), \( A^\beta_s \), \( s > 0, \beta > 0 \), is the Banach space of functions \( \varphi(x) = \sum_{|k| \geq 0} \exp(i k, x) \) of period \( 2\pi \) in each variable for which the norm
\[
\| \varphi \|_{s,\beta} = \sum_{|k| \geq 0} |\varphi_k| \exp\left(s|k|^{\beta}\right)
\]
is finite. It is obvious that \( A^\beta_s \subset G_{(1/\beta)} \), where \( G_{(1/\beta)} \) is the Gevrey class of 2\( \pi \)-periodic functions \( v(x_1, \ldots, x_p) \) of order \( 1/\beta \) of Beurling type [2, p. 113].

\( C^{(n,q)}(D) \) is the Banach space of functions \( u(t, x) \) with norm
\[
\| u \|_{C^{(n,q)}(D)} = \sum_{|k| \leq q} \max_{(t,x) \in D} \left| \frac{\partial^{j+|k|} u(t, x)}{\partial t^j \partial x_1^{k_1} \cdots \partial x_p^{k_p}} \right|.
\]

\( C^n([0, T], A^\beta_s) \) is the space of functions \( v(t, x) \) such that the partial derivatives \( \partial^j v/\partial t^j \), \( j = 0, n \), belong to the space \( A^\beta_s \) for each \( t \in [0, T] \) and are continuous in the norm of \( A^\beta_s \):
\[
\| v \|_{C^n([0, T], A^\beta_s)} = \max_{0 \leq p \leq n} \max_{t \in [0, T]} \left| \frac{\partial^p v}{\partial t^p} \right|_{s,\beta}.
\]

1. In the region \( D \) consider the following problem
\[
L(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}) u = \left( \frac{\partial}{\partial t} \right)^n u + \sum_{|k| \leq q} a_{j_0} \frac{\partial^{j+|k|} u}{\partial t^{j} \partial x_1^{k_1} \cdots \partial x_p^{k_p}} = 0,
\]
\[
B^0(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}) u \bigg|_{t=0} - B^1(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}) u \bigg|_{t=T} = \varphi(x),
\]
where
\[
B^j(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}) = \sum_{|k| \leq q} b^j_{j_0} \frac{\partial^{j+|k|} u}{\partial t^{j} \partial x_1^{k_1} \cdots \partial x_p^{k_p}};
\]


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\( b_{ij} = \text{col}(b_{ij}^1, \ldots, b_{ij}^n), b_{ij}^k \in \mathbb{C}^n, \ i = 0, 1, \ \varphi(x) = \text{col}(\varphi_1(x), \ldots, \varphi_n(x)); \ a_{j\nu} \in \mathbb{R}; \) and the operator \( L \) is Shilov-parabolic, i.e., the roots \( \lambda \) of the equation
\[
P(\lambda; \eta) \equiv \lambda^n + \sum_{|\nu| \leq q, j \leq n} a_{j\nu} i^{\nu_1} \eta_1^{\nu_1} \cdots \eta_n^{\nu_n} \lambda^j = 0
\] (3)

satisfy the following inequality for all \( \eta \in \mathbb{R}^p \):
\[
\max_{1 \leq j \leq n} \Re \lambda_j(\eta) \leq -c_1|\eta|^h + c_2; \quad c_1 > 0, \quad c_2 > 0, \quad h > 0.
\] (4)

The form of the domain \( D \) imposes periodicity conditions in the variables \( x_1, \ldots, x_p \) on the required solution \( u(t, x) \) and the functions \( \varphi_j(x), j = 1, n. \)

We seek a solution of the problem (1), (2) in the form of a series
\[
u(t, x) = \sum_{k \geq 0} u_k(t) \exp(ik, x),
\] (5)

where \( u_k(t) \) is a solution of the problem
\[
L\left(\frac{d}{dt}, ik\right)u_k(t) = 0, \quad B^0\left(\frac{d}{dt}, ik\right)u_k(t)\bigg|_{t=0} - B^1\left(\frac{d}{dt}, ik\right)u_k(t)\bigg|_{t=T} = \varphi_k,
\] (6)

and \( \varphi_k \) are the Fourier coefficients of the vector-valued function \( \varphi(x) \).

Assume that for all \( \eta = k \in \mathbb{Z}^p \) the roots \( \lambda_j(k) = \lambda_j, j = 1, n, \) of Eq. (3) are simple. Then the solution of the problem (6) can be expressed by the formula
\[
u_k(t) = \sum_{m=1}^n c_m(k) \exp(\lambda_m(k)t),
\] (7)

where the constants \( c_m(k) \) are defined by the system of equations
\[
\sum_{m=1}^n c_m(k) (B^0(\lambda_m, ik) - \exp(\lambda_m T) B^1(\lambda_m, ik)) = \varphi_k,
\] (8)

whose determinant has the form
\[
\Delta(k) \equiv \det \| B^0(\lambda_m, ik) - \exp(\lambda_m T) B^1(\lambda_m, ik) \|^n_{m=1}.
\] (9)

**Theorem 1.** A necessary and sufficient condition for uniqueness of the solution of the problem (1), (2) in the space \( \mathcal{C}^{(n, q)}(D) \) is that the following condition hold:
\[
\Delta(k) \neq 0, \quad k \in \mathbb{Z}^p.
\] (10)

**Proof.** This result follows from the uniqueness of the expansion of a periodic function in a Fourier series and is carried out using the plan of the proof of Theorem 4.5 of [4, Ch. 5].

2. Assume that the solution of the problem (1), (2) is unique. Then there exists a solution of the problem (6) for all \( k \in \mathbb{Z}^p \) having the form
\[
u_k(t) = \sum_{m=1}^n \Delta_m(k)|\Delta(k)|^{-1} \exp(\lambda_m(k)t),
\] (11)