ON THE LIPSCHITZ STABILITY OF COMPLEX SYSTEMS OF DIFFERENTIAL EQUATIONS

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Mathematical models of real physical processes and phenomena can be described by the so-called complex systems of differential equations. Among them are systems that decompose into interacting subsystems and intrasystemic connections. They have significant nonlinearities. In this connection the application of the existing methods of study of the dynamics of the behavior of such systems is quite difficult. We propose some approaches to the study of Lipschitz stability connected with the technique of applying Lyapunov functions.

1. Statement of the problem. Consider the system of differential equations

\[ \dot{x} = f(t, x), \]  

where the function \( f \) belongs to \( C[I \times \mathbb{R}^n; \mathbb{R}^n] \), \( I = [0, +\infty) \), \( (f(t, 0) \equiv 0) \), and \( f \) is such that the solution \( x(t; t_0, x_0) \) exists and is unique for any initial data \( x_0 \).

Definition [1]. The zero solution of the system (1) is Lipschitz stable if there exist constants \( \gamma > 1 \) and \( \delta > 0 \) such that

\[ |x(t; t_0, x_0)| < \gamma |x_0| \]

for any \( |x_0| < \delta \) and \( t \geq t_0 \geq 0 \).

Functions \( V : \mathbb{R}^+ \times \mathbb{R}^n \rightarrow \mathbb{R}^n \) for which the derivative over the system (1)

\[ \dot{V}(t, x) = \lim_{\alpha \rightarrow 0^+} \frac{1}{\alpha} \{ V(t + \alpha, x + \alpha f(t, x)) - V(t, x) \} \]

equals \( V(t, x) \) are said to be locally Lipschitzian with respect to \( x \).

For a system of the form (1) the following result is known [1]:

Proposition. Let \( f(t, x) \) in the system (1) be locally Lipschitzian with respect to \( x \). Then the zero solution of (1) is Lipschitz stable if and only if there exists a function \( V(t, x) \) defined for all \( x \) with \( |x| < \delta \) and \( t \geq 0 \) such that:

1. \( |x| \leq V(t, x) \leq L|x| \) (where \( L \) is a certain constant, \( L > 0 \));
2. \( |V(t, x) - V(t, y)| \leq |x - y| \) for any \( x, y \in \mathbb{R}^n \), \( t \geq 0 \), \( |x| < \delta \), \( |y| < \delta \);
3. \( \dot{V}(t, x) \leq 0 \).

If the system (1) is reduced to the form

\[ \dot{y} = \varphi(t, y) + g(t, y), \quad y(t_0) = y_0, \quad t_0 \geq 0, \]

where \( \varphi, g \in C[I \times S_\delta, \mathbb{R}^n] \), \( \varphi(t, 0) = g(t, 0) = 0 \), then the problem of stability of the zero solution of the system (1) can be solved as follows. Assume that the zero solution of the truncated system \( \dot{y} = \varphi(t, y) \) is Lipschitz stable (which can be established on the basis of the proposition) with Lipschitz constant \( M \). In addition suppose there exists a function

\[ \omega(t, u) \in C[I \times \mathbb{R}^+, \mathbb{R}^+], \quad \omega(t, 0) = 0, \]

that is nonincreasing, monotonic with respect to $u$ for any $t \in R^+$, and such that

$$|g(t, y)| \leq \omega(t, |y|).$$

If the zero solution of the scalar equation

$$\dot{u} = M\omega(t, u), \quad u(t_0) = u_0 \geq 0,$$

is Lipschitz stable, then the zero solution of the system (2) is also Lipschitz stable.

2. The main result. As noted above, for a large class of complex systems

$$\dot{x} = f(t, x),$$

defined in a certain domain

$$G : \{ t \geq 0; \quad 0 \leq \|x\| = \sqrt{(x, x)} < \infty\},$$

where $f(t, 0) \equiv 0$, $x \in R^n$, $f : I \times R^n \to R^n$, the following representation is possible:

$$\dot{x}_s = \varphi_s(t, x_s) + g_s(t, x), \quad s = 1, \ldots, k;$$

$$x_s \in R^{n_s}; \quad \varphi_s : I \times R^{n_s} \to R^{n_s}; \quad \sum_{s=1}^k n_s = n; \quad \dot{g}_s : I \times R^{n_s} \to R^n.$$

(4)

For the system (3) the systems of differential equations of the form

$$\dot{x}_s = \varphi_s(t, x); \quad \varphi_s(t, 0) = 0$$

are called isolated subsystems, and the vector $g(t, x) = \text{col}(g_1, \ldots, g_k)$ is called the interconnection vector of the isolated subsystems.

We shall assume that the vector-valued functions $f(t, x)$ and $\varphi_s(t, x_s)$, $s = 1, \ldots, k$, are such that the solutions $x(t; t_0, x_0)$ and $x_s(t; t_0, x_{s0})$, $s = 1, \ldots, k$, of the systems (3) and (5) respectively exist and are unique for any initial data $x_0$ and $x_{s0}$.

Suppose that for each isolated subsystem (5) of the original system (3) there exist functions $V_s(t, x_s)$ and $\xi_s(t, u_s)$ satisfying the following conditions:

$$\xi_s(t, u_s) \in C[I \times R^+; R], \quad (\xi_s(t, 0) \equiv 0);$$

$$S_{\delta_s} : \{ x \in R^{n_s}; \quad |x| < \delta_s \};$$

$$V_s(t, x_s) \in C[I \times S_{\delta_s}; R^+], \quad V_s(t, 0) \equiv 0.$$

In addition we shall assume that the functions $V_s(t, x_s)$ are locally Lipschitzian with respect to $x_s$ respectively and satisfy the condition

$$V_s(t, x_s) \geq \rho_s(|x_s|),$$

where $\rho_s(a) \in C[[0; \delta_s], R^+]$, $\rho_s(0) = 0$ is strictly monotonically increasing as $a$ increases, and is such that $\rho_s^{-1}(\alpha) \leq \alpha \cdot q_s(\alpha)$ ($q_s$ is a certain continuous function such that $q_s(\alpha) \geq 1$ for $\alpha \geq 1$, $s = 1, \ldots, k$). Here if the derivative over subsystem $s$ of (5) satisfies

$$\dot{V}_s(t, x_s) \leq \xi_s(t; V_s(t, x_s))$$

for $(t; x_s) \in I \times S_{\delta_s}$, and the zero solution of

$$\dot{u}_s = \xi_s(t, u_s), \quad (u_s(t_0) = u_{s0} \geq 0)$$

are called isolated subsystems, and the vector $g(t, x) = \text{col}(g_1, \ldots, g_k)$ is called the interconnection vector of the isolated subsystems.