ANALYSIS OF THE DISTRIBUTION OF ROOTS OF A POLYNOMIAL USING A GENERALIZED ROUTH SCHEME

A. T. Barabanov

UDC 62-50

We study the problem of determining the numerical characteristics of the distribution of the set of roots of a polynomial. The complete solution of this problem is achieved by applying a generalized Routh scheme, proposed previously for computing the Cauchy indices and the greatest common divisor of polynomials.

1. In previous papers [1, 2] we have proposed methods of computing the Cauchy index [3] of a rational function on the basis of generalized Routh-Hurwitz schemes. In the present paper we apply these results to the problem of the numerical characteristics of the distribution of the roots of a given polynomial.

To construct the algorithm for solving the problem we shall apply a table of numbers $c_{ik}, i = 1, \ldots, n_k; k = 1, \ldots, p,$ where the first two rows are given and the others are computed according to the Routh rule supplemented by a rule for shifting the elements of an auxiliary row. The transition from the two preceding rows $c_{i-2}, c_{i-1}, i = 1, 2, \ldots,$ to the following row $c_{i}, i = 1, 2, \ldots,$ is carried out after computing the elements of the auxiliary row

$$r_{ik} = c_{i-2} - \frac{c_{i-2}}{c_{i-1}}, \quad i = 1, 2, \ldots.$$  

Here $r_{1} = 0.$ In general $r_{2} = 0, r_{3} = 0, \ldots,$ but $r_{p+1} \neq 0.$ Then $c_{i} = (-1)^{m_i} r_{i-1}^{m_i-1}, i = 1, 2, \ldots.$ This is the shift rule.

The last row of the table ($k = p$) is the row after which a zero auxiliary row arises (all the numbers $r_{i}, i = 1, 2, \ldots,$ are zero). We now give an example of the construction of the table (rows 7 and 8 are needed in Sect. 4 below):

1. -2 -7 -6 0 0
2. 4 14 12 -2 -4
3. -1 -2 0 0 0 0 0 -1 -2
4. 6 12 -2 -4 0 0 6 12 -2 -4
5. 1/3 2/3 0 0 0 0 -1/3 2/3 0
6. 2 -4 0 0 0 0 0 -2 -4 0
7. 2 0 0 0 0 0 0 0 0 0
8. 4 0 0 0 0

Let $P(x_1, x_2, \ldots)$ be the number of changes of sign of the nonzero numbers in the sequence $x_1, x_2, \ldots.$ To apply the Routh table we shall use it to compute the two (Routh) numbers

$$R^- = P(l_1, \ldots, l_p) - R(c_1, \ldots, c_p), \quad (1)$$

where $l_k = (-1)^{n_k} c_k^{n_k}$ (here $n_k$ is the index of the last nonzero element in the $k$th row), and

$$R^+ = P(\beta_1, \ldots, \beta_p) - P(\alpha_1, \ldots, \alpha_p), \quad (2)$$

where $\alpha_k = \mu_k c_k, \beta_k = \mu_k c_k^{n_k}, \mu_k = -\mu_{k-2}(-1)^{m_k}, k = 3, 4, \ldots, \mu_1 = \mu_2 = 1.$

In our example

$$R^- = P(6, 4, -2, -4, 2/3, 4) - P(-2, 4, -1, 6, 1/3, 2) = -1;$$

$$R^+ = P(6, 4, 2, -2/3, -4) - P(2, -4, 1, -6, -1/3, -2) = -2.$$
In case it is necessary to indicate which initial data are processed to produce a Routh number, we shall write \( R^\pm(c_1, c_2) \), where \( c_1 = (c_1^1, c_2^1, \ldots, c_1^n) \), \( c_2 = (c_2^1, c_2^2, \ldots, c_2^n) \) are the notations for the initial rows. In what follows we shall also use the notation \( R^\pm(c_1/c_p) \), exhibiting the first and last rows of the table from which \( R^\pm \) is computed.

The main result of [1], which makes it possible to solve the problem we are studying here of determining the numerical characteristics of the distribution of roots of polynomials, can be written as the relations

\[
I_{-\infty}^0 A(x)/B(x) = R^-(c_1, c_2),
\]

\[
(-1)^{\sigma+1} I_{-\infty}^0 A(x)/B(x) = R^+(c_1, c_2),
\]

where \( A(x) \) and \( B(x) \) are given polynomials

\[
A(x) = a_0 x^s + a_1 x^{s-1} + \cdots + a_t x^{t-1}, \quad a_0 \neq 0, \quad a_t \neq 0, \quad 0 \leq t \leq s,
\]

\[
B(x) = b_0 x^m + b_1 x^{m-1} + \cdots + b_l x^{l-1}, \quad b_0 \neq 0, \quad b_l \neq 0, \quad 0 \leq l \leq m,
\]

\( \sigma = m + s \), and \( I_{-\infty}^0 A(x)/B(x) \) is the Cauchy index [2] of the rational function \( f(x) \) with respect to the open interval \((c, \beta)\), (i.e., \( \sum \text{sgn} f(x_i + 0) \) over all poles \( x_i \) at which \( f(x) \) changes sign).

In the computation of the Routh numbers in (3) and (4) the initial pair of rows is defined by the relations

1) \( c_1^1 = b_{t-1}, \quad c_2^1 = (-1)^{\sigma+1} a_{t-1}, \quad m - s - 1 \geq 0; \)
2) \( c_1^1 = (-1)^{\sigma+1} b_{t-1}, \quad c_2^1 = a_{t-1}, \quad m - s - 1 < 0. \)

If the initial pair is defined as the rows \( a = (a_0, a_1, \ldots, a_t), \quad b = (b_0, b_1, \ldots, b_l) \), i.e., by direct differentiation of the polynomials, we shall have

\[
(-1)^{\sigma+1} I_{-\infty}^0 A(x)/B(x) = R^-(a, b) \quad I_{-\infty}^0 A(x)/B(x) = R^+(a, b).
\]

This is because of the property of the index \( I_{-\infty}^0 (-1)^n R(x) = (-1)^n I_{-\infty}^0 R(x) \) for every integer \( n \) (multiplying one of the polynomials by \( (-1)^{\sigma+1} \), we thereby pass to the rows \( a, b \)). One may also note the following relation:

\[
R^{\pm}(pc_1, qc_2) = (\text{sgn} pq) R^{\pm}(c_1, c_2)
\]

for all nonzero numbers \( p \) and \( q \).

Finally, we note that the construction of the Routh table of numbers \( c_i^k, i = 1, \ldots, n_k, \quad k = 1, \ldots, p \), in this way makes it possible to determine [1] the GCD of the polynomials \( A(x) \) and \( B(x) \) up to a trivial factor (of the form \( cx^r \)). It is determined by the last row of the table

\[
C(x) = c_1^p x^{n_p-1} + c_2^p x^{n_p-2} + \cdots + c_k^n.
\]

The case where \( A(x) = B'(x) = dB(x)/dx \) in relations (3) and (4) will be especially important below. In this case the second row of the Routh scheme is completely determined by the rule for differentiation. At the same time we shall need to indicate the last row of the Routh scheme in computing such an index and the corresponding Routh number. Denoting the first and last rows as \( c_1 \) and \( c_p \), instead of (3), (4) we have

\[
I_{-\infty}^0 B'(x)/B(x) = R^-(c_1/c_p),
\]

\[
I_{-\infty}^0 B'(x)/B(x) = R^+(c_1/c_p),
\]

since \( \sigma \) is an odd number in this case.

2. In the problem of the numerical characteristics of the distribution of roots of a polynomial it is necessary to answer the question of the number of roots of one class or another—right (belonging to the open right