ASYMPTOTIC EXPANSIONS FOR SOLUTIONS OF n-TH ORDER LINEAR ORDINARY DIFFERENTIAL EQUATIONS WITH TWO TURNING POINTS

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Asymptotic expansions for solutions of n-th order linear differential equation with two turning points are constructed in Olver's form. Analytic properties of the coefficients of the series obtained are investigated.

Bibliography: 7 titles.

A previous treatment of problems concerning asymptotic expansions for the solutions of second-order differential equations with two turning points is presented in [1-4]. Expansions in Olver's form [1] or in Cherry's form [2] are constructed. The asymptotic nature of the formal expansions obtained is investigated both in the case of real [1] and complex [3] variables. We shall study the extension of this problem to n-th order differential equations. In the present paper we develop the methods used by M. V. Fedoryuk [5, 6] for the problem of n-th order differential equations without turning points.

1. Consider the differential equation

$$ly = e^ny^{(n)} + \sum_{j=1}^{n} e^{n-j}q_j(z, \lambda)y^{(n-j)} = 0.$$  (1)

Here $z$ is a complex variable ranging over a domain $D$, $\lambda$ is a complex parameter that ranges over a domain $G (0 \in G)$, and $e$ is a “small” positive parameter. It is assumed that the functions $q_j(z, \lambda)$ are holomorphic in $\Omega = D \times G$. For the symbol of the operator $l$ we introduce the notation

$$l(z, p, \lambda) = p^n + \sum_{j=1}^{n} q_j(z, \lambda)p^{(n-j)}.$$  

In accordance with [5], the point $z_0$ is called a turning point for Eq. (1) provided the equation $l(z, p, \lambda) = 0$ has multiple root w.r.t. $p$. The point $z_0$ is called a simple turning point provided that 1) the equation $l(z, p, \lambda) = 0$ has one root $p_0$ of multiplicity two, other roots are simple; 2) $l_z(z_0, p_0, \lambda) \neq 0$.

We assume that: the symbol $l(z, p, \lambda)$ can be represented in the form

$$l(z, p, \lambda) = (p^2 - 2a(z, \lambda)p + \alpha(z, \lambda))(p - p_3(z, \lambda)) \cdots (p - p_n(z, \lambda)),$$  (2)

and the roots of the first bracket, $p_{1,2}(z, \lambda) = \alpha(z, \lambda) \pm \sqrt{F(z, \lambda)}$, $F(z, \lambda) = \alpha^2 - \beta$, coincide for $z = z_j, j = 1, 2; F(z_j, \lambda) = 0, F_z(z_j, \lambda) \neq 0 (\lambda \neq 0)$. We also assumed that the roots $p_3(z, \lambda), \ldots, p_n(z, \lambda)$ are distinct and different from $p_{1,2}(z, \lambda)$ for $(z, \lambda) \in \Omega$; the two $z_1$ and $z_2$ tend to $z = 0$ as $\lambda$ tends to $0$, i.e., $z_1(0) = z_2(0) = 0$. Hence the points $z_1(\lambda)$ and $z_2(\lambda)$ are simple turning points for Eq. (1).

Because of the fact that $q_j(z, \lambda)$ are holomorphic in $\Omega$, the functions

$$a(z, \lambda), \beta(z, \lambda), p_3(z, \lambda), \ldots, p_n(z, \lambda)$$

are also holomorphic in $\Omega$. The points $z_1(\lambda)$ and $z_2(\lambda)$ are branch points of the second order for the functions $p_1(z, \lambda)$ and $p_2(z, \lambda)$ in $\Omega$.

By using the procedure in [6], we obtain $(n - 2)$ formal linear independent asymptotic solutions of equation (1) of the form

$$A_j(z, \lambda, e) \exp \left( \frac{1}{e} \int_{z_0}^{z} p_j(t) dt \right), \quad A_j(z, \lambda, e) = \sum_{n=0}^{\infty} A_{jn}(z, \lambda)e^n,$$

where \( A_{j_0}(z, \lambda) = \exp \left( -\frac{1}{2} \int_{z_0}^{z} p_j(t, \lambda) \frac{\mu(t, \lambda)}{\nu_p(t, \lambda)} \, dt \right), \quad j = 3, \ldots, N. \)

To obtain an asymptotic representation for the rest two linear independent solutions of Eq. (1), we use the transformation

\[
y(z, \lambda, \epsilon) = \exp \left( \frac{1}{\epsilon} \int_{z_0}^{z} \alpha(t, \lambda) dt \right) \tilde{y}(z, \lambda, \epsilon).
\]

Then for the functions \( \tilde{y}(z, \lambda, \epsilon) \) we get the equation

\[
(l_0 + \epsilon l_1 + \epsilon^2 l_2 + \cdots) \tilde{y}(z, \lambda, \epsilon) = 0,
\]

where the operator \( l_0 \) has the symbol

\[
l_0(z, p, \lambda) = (p^2 - F(z, \lambda))(p - \hat{p}_0(z, \lambda)) \cdots (p - \hat{p}_n(z, \lambda)); \quad \hat{p}_j \equiv p_j - \alpha
\]

and the symbol of the operator \( l_1 \) is of the form

\[
l_1(z, p, x) = \alpha' \sum_{j, l = 1}^{n} \prod_{k \neq l} (p - \hat{p}_k).
\]

We seek two asymptotic solutions of Eq. (2) corresponding to the multiplier \((p^2 - F)\) in the form

\[
y_{1,2}(z, \lambda, \epsilon) = A(z, \lambda, \epsilon) D^{(k)}(z, \lambda, \epsilon) + B(z, \lambda, \epsilon) D_{n}^{(k)}(z, \lambda, \epsilon).
\]

Here \( D_{n}(x) \) is a solution of the Weber differential equation [7]

\[
\frac{d^2 D}{dz^2} + \left( \nu + \frac{1}{2} - \frac{z^2}{4} \right) D = 0
\]

and

\[
A(z, \lambda, \epsilon) = \sum_{n \geq 0} a_n(z, \lambda) \epsilon^n, \quad B(z, \lambda, \epsilon) = \sum_{n \geq 0} b_n(z, \lambda) \epsilon^n; \quad \tau(\lambda, \epsilon) = \sum_{n \geq 0} \tau_n(\lambda) \epsilon^n.
\]

The function \( \xi(z, \lambda) \) and coefficients \( a_n(z, \lambda), b_n(z, \lambda), \tau_n(\lambda) \) are to be determined to ensure that \( \xi(z, \lambda), \) \( a_n(z, \lambda), \) and \( b_n(z, \lambda) \) be analytic functions in \( \Omega \) and \( \xi(z, \lambda) \neq 0. \) We substitute expansions (4) into Eq. (2) and single out the leading terms w.r.t. \( \epsilon. \) For this purpose we use the leading terms w.r.t. \( \epsilon \) in the representation of the derivatives \( D_{n}^{(k)}(x) \) via \( D_{n}(x) \) and \( D_{n}^{(k)}(x) \) by means of the Weber equation

\[
D_{n}^{(k)}(x) = (-i)^n e^{-n}(\xi^2/4 - \tau_0)nD_{n}(x)[1 + O(\epsilon)] + (-i)^{n-1} \times
\]

\[
x e^{-n-1}(\xi^2/4 - \tau_0)n^{-2} \frac{d}{dx} \left( \frac{\xi^2}{4} - \tau_0 \right)n(n-1)D_{n}(x)[1 + O(\epsilon)];
\]

\[
D_{n}^{(k+1)}(x) = (-i)^n e^{-n}(\xi^2/4 - \tau_0)nD_{n}(x)[1 + O(\epsilon)] + \frac{d}{dx} \left( \frac{\xi^2}{4} - \tau_0 \right)n^{-3} \frac{d}{dx} \left( \frac{\xi^2}{4} - \tau_0 \right)(n-2)D_{n}(x)[1 + O(\epsilon)].
\]

Let \( l_0^{(k)} \) and \( l_0^{(0)} \) be even and odd parts of the symbol \( l_0: \) \( l_0^{(k)} = \frac{1}{2} [l_0(z, p, \lambda) + l_0(z, -p, \lambda)], \) \( l_0^{(0)} = \frac{1}{2} [l_0(z, p, \lambda) - l_0(z, -p, \lambda)]. \)

In the first approximation w.r.t. \( \epsilon \) we get the system

\[
A \begin{pmatrix} a_0 \\ b_0 \end{pmatrix} \equiv \begin{pmatrix} l_0^{(k)}(z, f^{1/2} \xi, \lambda), & e^{-\xi/4} f^{1/2} \xi_0^{(0)}(z, f^{1/2} \xi, \lambda) \\ e^{\xi/4} f^{-1/2} \xi^{(0)}(z, f^{1/2} \xi, \lambda), & l_0^{(k)}(z, f^{1/2} \xi, \lambda) \end{pmatrix} \begin{pmatrix} a_0 \\ b_0 \end{pmatrix} = 0,
\]