THE CLASSES $B_{m,1}$ AND HÖLDER CONTINUITY FOR DOUBLY DEGENERATE PARABOLIC EQUATIONS

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Inner and boundary Hölder estimates for nonnegative weak solutions of quasilinear doubly degenerate parabolic equations are established. The proof of these results is based on studying some classes $B_{m,1}$ that can be considered as extensions of the classes $B_2$ introduced by Ladyzhenskaya and Uraltseva and the classes $B_m$ introduced by DiBenedetto. The embedding of the classes $B_{m,1}$ in appropriate Hölder spaces is proved. Bibliography: 20 titles.

1. INTRODUCTION

Consider a quasilinear equation of the form
$$\frac{\partial u}{\partial t} - \partial / \partial x_i a^i(x, t, u, \nabla u) + b(x, t, u, \nabla u) = 0$$

in the cylinder $Q_T = \Omega \times (0, T]$, $\Omega \subset \mathbb{R}^n$, $n \geq 1$, where $x = (x_1, \ldots, x_n)$, $\Omega$ is a bounded open set, $\nabla u$ denotes the spatial gradient of the function $u(x, t)$. The functions $a^i(x, t, u, p)$ and $b(x, t, u, p)$ satisfy the Caratheodory conditions and the following inequalities:

$$a_i(x, t, u, p) \geq \gamma_0 |u|^{\alpha} |p|^m - \varphi_0(x, t), \quad \gamma_0 > 0,$$

$$|a^i(x, t, u, p)| \leq \mu_1 |u|^{\alpha} |p|^{m-1} + |u|^{\gamma} \varphi_1(x, t), \quad \alpha = l/m,$$

$$|b(x, t, u, p)| \leq \mu_2 |u|^{m'} |p|^m + \varphi_2(x, t), \quad 1/m + 1/m' = 1,$$

where $m \geq 2$, $l \geq 0$, and $\varphi_i(x, t)$ are given nonnegative functions subject to the following requirements:

$$\varphi_0, \varphi_1^m, \varphi_2 \in L_{q_0}(Q_T)$$

with appropriate exponents $q$ and $q_0$.

A typical example of the equations under consideration is
$$\frac{\partial u}{\partial t} - \partial / \partial x_i \{a_0 |u|^{\alpha} \nabla u|^{m-2} \partial u / \partial x_i\} = 0, \quad a_0 > 0, \quad m \geq 2, \quad l \geq 0.$$ (1.4)

The following equations are special cases of (1.4):
$$\frac{\partial u}{\partial t} - \partial / \partial x_i \{a_0 |u|^{m} \nabla u|^{m-2} \partial u / \partial x_i\} = 0, \quad a_0 > 0, \quad m = 2, \quad l = 0,$$

i.e., the heat equation;

$$\frac{\partial u}{\partial t} - \partial / \partial x_i \{a_0 |u|^{\alpha} \nabla u|^{m-2} \partial u / \partial x_i\} = 0, \quad m = 2, \quad l > 0,$$

i.e., the porous medium equation or the equation of Newtonian polytropic filtration;

$$\frac{\partial u}{\partial t} - \partial / \partial x_i \{a_0 |\nabla u|^{m-2} \partial u / \partial x_i\} = 0, \quad m > 2, \quad l = 0,$$

i.e., the $m$-Laplacian parabolic equation or the equation of non-Newtonian elastic filtration;
for \( m > 2, l > 0 \), Eq. (1.4) is known as the equation of non-Newtonian polytropic filtration. This equation is a doubly degenerate equation because it has a singularity both for \( u = 0 \) and for \( \nabla u = 0 \).

The regularity of weak solutions of quasilinear parabolic equations has been intensively studied during the last thirty years. The first question which naturally arises in the regularity problem is that concerning the Hölder estimates for weak solutions of the equations considered.

Well-known results by Ladyzhenskaya–Uraltseva [1] in the 1960's concerned only uniformly parabolic equations (i.e., Eqs. (1.1)-(1.3) in the case \( m = 2, l = 0 \)).

The regularity of weak solutions of quasilinear degenerate parabolic equations has been investigated only recently. Chen Ya-Zhe [2], DiBenedetto and Friedman [3], and A. V. Ivanov [4], by means of different methods and for different subclasses of equations of the form (1.1)–(1.3), have obtained the Hölder estimates of weak solutions in the case \( m = 2, l > 0 \).

An important contribution was made by DiBenedetto [5] who established Hölder estimates for weak solutions of equations of the form (1.1)–(1.3) in the case \( m = 2, l = 0 \). Chen Ya-Zhe and DiBenedetto [6] have completed this result by obtaining similar estimates in the case \( 1 < m < 2, l = 0 \).

The Hölder estimates for weak solutions of multidimensional \( (n > 1) \) quasilinear doubly degenerate parabolic equations (i.e., in the case \( m > 2, l > 0 \)) were first obtained by the author [7]–[10]. In [7]–[10] I was concerned only with positive weak solutions of Eqs. (1.1)–(1.3) and obtained uniform inner and boundary Hölder estimates independent of their infimums. Such estimates were used in [11]–[13] for proving the existence of nonnegative Hölder-continuous weak solutions of the Cauchy–Dirichlet problem under conditions (1.2) and (1.3) and some additional assumptions (in particular, under the assumption of a more limited growth of the function \( b(x, t, u, p) \)).

In this paper, I establish inner and boundary Hölder estimates directly for nonnegative weak solutions. The proof of these results is based on the study of some classes \( B_{m,l} \) which can be considered as extensions of the classes \( B_2 \) introduced by Ladyzhenskaya–Uraltseva [1] and the classes \( B_m \) introduced by DiBenedetto [5] and on establishing the embeddings of these extensions in appropriate Hölder spaces. It should be said that in the case \( n = 1 \) Esteban and Vazquez obtained Hölder estimates of weak solutions of the Cauchy problem for equation (1.4) earlier [14] in a different way.

In the case \( m > 2, l > 1 \) the Hölder continuity of weak solutions of Eqs. (1.1)–(1.3) is the best possible smoothness of these solutions. Consider the following counterexample. The function

\[
    u(x, t) = t^{-\alpha \beta} \left[ 1 - c \left( \frac{|x|}{t^{\alpha \beta}} \right)^{\gamma} \right], \quad x = (x_1, \ldots, x_n),
\]

is a weak solution (in the sense defined below) of Eq. (1.4) in \( \mathbb{R}^n \times (\varepsilon, T], \varepsilon > 0, T > \varepsilon, \) if \( m + l - 2 > 0 \) and

\[
    \alpha^{-1} = n(l + m - 2) + m, \quad \beta = n, \quad \gamma = m/(m - 1), \quad \delta = (m - 1)/(l + m - 2),
\]

and for an appropriate constant \( c \).

In the case \( l > 1 \) the spatial gradient of function (1.5) is not bounded (because in this case \( \delta - 1 = (1 - l)/(l + m - 2) < 0 \)). It is easy to see that for \( l > 1 \) the function (1.5) satisfies the Hölder condition in \( x \) with exponent \( \lambda = (m - 1)/(m + l - 2) \in (0, 1) \) and this assertion fails to be true for any \( \lambda' > \lambda \). Hence the best possible regularity of weak solutions of equations of the form (1.1)–(1.3) in the case \( l > 1 \) is the Hölder continuity with the exponent \( \lambda = (m - 1)/(m + l - 2) \).

Basic notation:

\[
    |\Omega| = \text{mes}_n \Omega, \quad \Omega \subset \mathbb{R}^n; \quad |\mathcal{Q}| = \text{mes}_{n+1} \mathcal{Q}, \quad \mathcal{Q} \subset \mathbb{R}^{n+1};
\]

\[
    B_{\rho}(x_0) = \{ x \in \mathbb{R}^n : |x - x_0| < \rho \}; \quad Q(\rho, r; x_0, t_0) = B_{\rho}(x_0) \times [t_0 - r, t_0];
\]

\[
    (u - k)^+ = \sup(u - k, 0), \quad (u - k)^- = \sup(k - u, 0), \quad a^+ = \sup(a, 0);
\]

\[
    \sup(f, D) \equiv \sup_{y \in D} f(y), \quad \inf(f, D) \equiv \inf_{y \in D} f(y), \quad \text{osc}(f, D) = \sup(f, D) - \inf(f, D);
\]

\[
    \mathcal{C}^1(Q_T) = \{ u \in \mathcal{C}^1(Q_T) : u = 0 \text{ on } S_T \}, \quad S_T = \partial \Omega \times (0, T];
\]

\[
    \mathcal{C}^0_0(Q_T) = \{ u \in \mathcal{C}^0(Q_T) : u = 0 \text{ on } \Gamma_T \}, \quad \Gamma_T = S_T \cup \{ \partial \Omega \times \{ t = 0 \} \};
\]

\[
    V_m(Q_T) = \text{L}^\infty(0, T; L_m(\Omega)), \quad W^{m,0}_m(Q_T) = L_m([0, t]; W^m_m(\Omega)), \quad W^m_m(\Omega) = H^m_m(\Omega), \quad \left\| u \right\|_m^{m, Q_T} = \sup_{t \in [0, T]} \left\| u(x, t) \right\|_m + \left\| \nabla u \right\|_m^{m, Q_T},
\]

\[
    \left\| u \right\|_{m, \Omega} = \left\| u \right\|_{L_m(\Omega)}, \quad \left\| u \right\|_{m, Q_T} = \left\| u \right\|_{L_m(Q_T)}.
\]