ON NEW ESTIMATES FOR THE NAVIER–STOKES EQUATIONS AND
GLOBALLY STABLE ATTRACTORS

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For the two-dimensional Navier–Stokes equations and for a number of their globally stable approximations (the
Galerkin–Faedo method, the Galerkin–Faedo method discrete in time, the implicit finite-difference methods
(19i)) the author presents new a priori estimates which prove the existence of a compact minimal global B-
attractor for the Navier–Stokes equations (this fact was first proved by the author in 1972, see [1]) and also for
the approximations mentioned above. Similar results for many problems of the theory of viscous incompressible
fluids and continuum mechanics are valid. Bibliography: 18 titles.

Twenty years have passed since the publication of the paper [1] in which the existence of a compact
minimal global B-attractor \( M \) was proved for the two-dimensional Navier–Stokes equations (let us shortly
call such attractors MGA and denote by the letter \( M \) with some indices). For terminology, see [2, 3]. In
the 1980s many papers were devoted to this object (e.g., see the bibliography on attractors [4] compiled by
G. R. Sell). In resent years investigations on the stability of the attractors for various approximations to the
problem were started. In [5] I performed such an investigation for finite-difference and difference-differential
approximations to semilinear equations of the parabolic type. But a wish to carry out similar analysis for
the Navier–Stokes equations required new a priori estimates of their solutions and a new method of proof
suitable for the approximations. Let us explain this in more detail.

The existence of a compact \( M \) is caused by the following two properties of the problem: 1) the existence
of an absorbing set in the phase space \( H_0 \) and 2) the compactness (in \( H_0 \)) of the solution operators \( V_t \) of
the problem for \( t > 0 \).

The first property can be derived from the energy relation. The second was derived in [1] from the
integral relation obtained by the inner multiplication in \( L_2(\Omega, \mathbb{R}^2) \) of the Navier–Stokes equations for \( v(t) \)
by \( \Delta v(t) \), the value of the Stokes operator \( \Delta \) on \( v(t) \). Denote this relation by the star (\(*\)). Moreover,
substantial use was made of the Solonnikov–Cattabriga inequality [6–9] for the operator \( \Delta \), which holds in
\( \Omega \) with sufficiently smooth boundary \( \partial \Omega \) (e.g., \( \partial \Omega \subset C^2 \)).

For globally stable approximations properties (1) and (2) should be preserved. The first property was
usually preserved because a well-defined analog of the energy relation was retained. A different situation
arises with regard to property (2). Its validity followed from [1] for two kinds of approximations (if \( \partial \Omega \subset C^2 \): for
the Rothe approximations and Galerkin–Faedo (G-F) approximations, with the eigenfunctions of \( \Delta \) as
coordinate functions. (For them the relation (\(*\)) does hold.) But these approximations are interesting
mostly from a theoretical point of view. For the other approximations either (\(*\))-type relations are missing
or the boundaries of the domains where the approximations are determined are not smooth enough. So it
was necessary to look for other proofs of the compactness of the solution operators for the Navier–Stokes
equations and their approximations. I have done this by using, instead of the Navier–Stokes equations,
only the three integral relations (3), (8), (9) (which hold for many approximations) and of the imbedding
theorems, only the inequality

\[
\|u\|_{4, \Omega}^4 \leq \frac{1}{2} \|u\|_{2, \Omega}^2 \|\partial_\Omega u\|_{2, \Omega}^2
\]

(1)

that holds for any \( u \in W^4_2(\Omega) \) in any domain \( \Omega \subset \mathbb{R}^2 \).

The suggested method of obtaining the estimates is applicable to many problems and their approximations:
1) to the modifications of the three-dimensional Navier–Stokes equations offered in [10] (see also [11]),
2) to the Navier–Stokes equations themselves under nonhomogeneous boundary conditions \( v|_{\partial \Omega} = a|_{\partial \Omega} \) with
\( a = \text{rot } b, b \in W^2_2(\Omega) \) if \( \partial \Omega \) is piecewise smooth, 3) for the heat-convection equations and the equations of


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the magnetohydrodynamics of viscous incompressible fluids, and also for a number of other equations of the mechanics of continua.

In this publication I will formulate a number of results concerning the two-dimensional Navier–Stokes equations reported at the conferences [12–14] and partially described in papers [14, 15]. For brevity, I assume everywhere that \( \Omega \) is a bounded domain in \( \mathbb{R}^2 \), not always mentioning this explicitly. No limitations are laid on \( \partial \Omega \).

1. Properties of the Solutions to the Navier–Stokes Equations

We consider the problem

\[
\begin{align*}
\partial_t v(t) - \nu \Delta v(t) + v(t) \cdot \nabla v(t) &= -\nabla p(t) + f(t), \\
\text{div} \, v(t) &= 0, \\
v(t)|_{\partial \Omega} &= 0, \\
v|_{t=0} &= \varphi,
\end{align*}
\]

where \( v(t): \Omega \rightarrow \mathbb{R}^2 \) and \( p(t): \Omega \rightarrow \mathbb{R}^1 \) are unknown functions,

\[
v = \text{const} > 0, \quad v(t) \cdot \nabla v(t) = \sum_{k=1}^2 v_k(t) \partial_{x_k} v(t),
\]

and \( v_k(t) \) are Cartesian components of \( v(t) \). As the phase space, we take \( J(\Omega) \), but denote it by the letter \( H_0 \). The inner product in \( H_0 \) will be denoted by \( (\cdot, \cdot) \) and the norm by \( \| \cdot \| \) (as in \( L_2(\Omega, \mathbb{R}^2) \)). \( H_0 \) is the closure of \( J(\Omega) \) in \( L_2(\mathbb{R}^2) \) [8]. The closure of \( J(\Omega) \) in the norm of the Dirichlet integral will be denoted by \( H_1 \) (this is the space \( H(\Omega) \) of [8], see Ch. 1, §2). The scalar product of the elements \( u \) and \( v \) in \( H_1 \) is written as \( (u, v)_{H_1} = (\partial_x u, \partial_x v) \) and the norm as \( \| u \|_{H_1} = \| \partial_x u \| \). By \( H_{-1} \) we designate the space dual to \( H_1 \) with respect to \( H_0 \), and by \( \| \cdot \|_{-1} \) the norm in it.

Throughout the paper we assume that \( \varphi \in H_0 \) and \( f \in \mathcal{F}(T) \). The latter means the following: \( f(t) \in H_{-1} \) for a.a. \( t \in [0, T] \) and \( (\int_0^T \| f(t) \|_{-1}^2 \, dt)^{1/2} = \| f \|_{\mathcal{F}(T)} \) exists and is finite. First, let us formulate a known statement ([8, 1]).

**Theorem 1.** For any \( \varphi \in H_0 \) and \( f \in \mathcal{F}(T) \), problem \( (2k) \), \( k = 1, 2 \), has a unique solution \( v(t, \varphi, f) \) with the following properties: \( v \in C([0, T], H_0) \cap L^2(\mathbb{R}^2) \) and for almost all \( t \in [0, T] \) it satisfies the energy relation

\[
\frac{1}{2} \frac{d}{dt} \| v(t, \varphi, f) \|^2 + \nu \| \partial_x v(t, \varphi, f) \|^2 = (f(t), v(t, \varphi, f)).
\]

Moreover, we have the estimates

\[
\| v \|_{Q_T} \leq \Phi_1(t, \| \varphi \|, \| f \|_{\mathcal{F}(T)}, \nu^{-1}),
\]

where

\[
\| v \|_{Q_T} = \text{ess sup}_{t \in [0, T]} \| v(t) \| + \| \partial_x v \|_{L^2(Q_T)},
\]

and \( \Phi_1(\ldots) \) (like all subsequent functions \( \Phi_k(\ldots) \)) is a continuous, monotonically nondecreasing function of the indicated arguments (all \( \Phi_k(\ldots) \) can be calculated explicitly). The solution \( v(t, f, \varphi) \), as an element of \( H_0 \), depends continuously on \( (t, f, \varphi) \in [0, T] \times \mathcal{F}(T) \times H_0 \). The solution operators \( V(t, f): \varphi \in H_0 \rightarrow v(t, f, \varphi) \in H_0 \) are uniformly differentiable on any ball \( B_R(\Omega) = \{ u | u \in H_0, \| u \| \leq R \} \). If \( f \) belongs to \( \mathcal{F}(T) \) but is independent of \( t \), i.e., if \( f \in H_{-1} \), then the solution operators \( V(t, f) \equiv V_t(f) \) form a semigroup \( \{ V_t(f), t \in \mathbb{R}^+ \} \). This semigroup is continuous, and the balls \( B_{R}(H_0) \), where \( R > R_0 = \lambda_1^{-1/2} \| f \|_{-1} \), are \( B \)-absorbing sets.

For the solutions \( v(t) \equiv v(t, f, \varphi) \), the following estimates hold:

\[
\begin{align*}
\| v(t) \|^2 &\leq \| v(0) \|^2 e^{-\lambda_1 t} + (R_0^2)^2(1 - e^{-\lambda_1 t}), \\
\| v(t) \| &\leq \max\{ \| v(0) \|, R_0\}, \\
\int_0^t \| \partial_x v(t) \|^2 \, dt &\leq \| v(0) \|^2 + t \| f \|_{-1}^2.
\end{align*}
\]

Estimates (4) and (5) are derived from (3).

Now let us formulate new results.

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