ASYMPTOTICALLY PROPER CONSTANTS IN THE LYAPUNOV THEOREM

G. P. Chistyakov

A new asymptotic representation for the distribution function of the normalized sums of not necessary identically distributed random variables in the Lyapunov CLT is established. Asymptotically proper constants in the Berry–Esseen inequality are obtained. Bibliography: 13 titles.

We consider independent random variables $X_1, X_2, \ldots, X_n$ with zero means, positive variances $\sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2$, and finite third moments $\beta_1, \beta_2, \ldots, \beta_n$. Let $F_1, F_2, \ldots, F_n$ denote their distribution functions, and let $F^3$ be the class of nondegenerate distribution functions that have the third moment and zero mean.

Set $\sigma^2 = \sigma_1^2 + \cdots + \sigma_n^2$, $\beta = \beta_1 + \cdots + \beta_n$, and $L = \beta/\sigma^3$. By the Lyapunov theorem, if $L \to 0$, then the distribution functions $F$ of the standardized sums $S_n = (X_1 + \cdots + X_n)/\sigma$ converge to the standard normal distribution function $\Phi$ in the uniform metric $\rho$. Moreover, by the Berry–Esseen result, there exists an absolute constant $c_0$ such that

$$\rho(F, \Phi) \leq c_0 L. \tag{1}$$

Kolmogorov [1] attached great importance to finding the smallest value $c_1$ of the constant $c_0$. He conjectured that $c_1 = 1/\sqrt{2\pi}$. Esseen proved that the constant $c_0$ in inequality (1) cannot be less than $(\sqrt{10} + 3)/(6\sqrt{2\pi})$. This is a corollary from the following theorem.

**Theorem A (Esseen [2]).** Let the summands $X_j$ in the sums $S_n$ be identically distributed with distribution function $F_1$. Then

$$A \overset{\text{def}}{=} \sup_{F_1 \in F^3} \frac{\sigma^3}{\beta_1} \lim_{n \to \infty} \sup_n \rho(F, \Phi) = \frac{\sqrt{10} + 3}{6\sqrt{2\pi}}. \tag{2}$$

At the same time, the extreme value of $A$ is attained at a two-point lattice distribution function with jumps $(\sqrt{10} - 2)/2$ and $(4 - \sqrt{10})/2$ at the points $x_1 = -h(4 - \sqrt{10})/2$ and $x_2 = h(\sqrt{10} - 2)/2$, respectively, where $h > 0$.

Rogozin made Esseen's result more precise. Let $N$ be the set of all normal distribution functions, and put $\rho(F, N) = \inf_{Q \in N} \rho(F, Q)$.

**Theorem B (Rogozin [3]).** Let the summands $X_j$ in the sums $S_n$ be identically distributed with distribution function $F_1$. Then

$$A \overset{\text{def}}{=} \sup_{F_1 \in F^3} \frac{\sigma^3}{\beta_1} \lim_{n \to \infty} \sup_n \rho(F, N) = \frac{1}{\sqrt{2\pi}}. \tag{3}$$

The extreme value of $A$ is attained at a symmetric Bernoulli distribution function.

The exact value of the constant $c_1$ has not been found up to now, though a lot of papers are devoted to this problem. The subject is described in detail in [4], and we refer our reader to this work. The problem of constants in other metrics is discussed in [5–7]. In particular, the $L_p$-metrics ($1 \leq p < \infty$) are considered there.

Esseen, Rogozin, Zolotarev, and others relied in their research on an asymptotic representation of the distribution function $F$, which was established by Esseen.

Theorem C (Esseen, [8]). Let the summands $X_j$ in the sums $S_n$ be identically distributed with distribution function $F_1 \in \mathcal{F}^3$. Then uniformly in $x \in \mathbb{R}$ as $n \to \infty$ we have

$$F(x) = \Phi(x) + \frac{\alpha_1}{6 \sigma_1^2} \frac{1}{\sqrt{2 \pi n}} (1 - x^2) e^{-x^2/2} + \frac{h}{\sigma_1 \sqrt{2 \pi n}} e^{-x^2/2} \Psi_n(x) + o \left( \frac{1}{\sqrt{n}} \right),$$

(2)

where $\Psi_n(x) = \frac{1}{2} - \{(x \sqrt{n} - \frac{a_1}{\sigma_1}) \frac{2}{\sqrt{n}} \}$, and $\{x\}$ is the fractional part of $x$. For the lattice function $F_1$, the parameters $a$ and $h$ determine the minimal lattice $x_m = a + mh$ ($m = 0, \pm 1, \pm 2, \ldots$) of $F_1$-probability equal to one. In this case, $h > 0$. If $F_1$ is not a lattice function, then we assume that $h = h \Psi_n(x) = 0$.

Note that the remainder term in expansion (2) does not always have the property of uniformity over the class $\mathcal{F}^3$. An asymptotic representation in the form (2) was obtained for differently distributed random variables $X_j$ under additional requirements to these variables. No analog of representation (2) has been known in the general case. See [5–7] for details on the problem of asymptotic representations and the results obtained.

Apart from the problem of finding the value of the constant $c_1$, Kolmogorov [1] was also interested in finding exact estimates of the remainder terms and improving the approximation in the Lyapunov theorem.

We use the notation $\rho(l, x) = \sup |F(x) - \Phi(x)|$ and $\rho(l) = \sup \rho(F, \Phi)$, where the supremum is taken over all distribution functions $F_1, \ldots, F_n$ such that $L = l$.

We will calculate the constant $c^* = \lim \sup \frac{1}{l} \rho(l)$ and the function $c^*(x) = \lim \sup \frac{1}{l} \rho(l, x)$. A similar question arises for the constant $c^*_1$ and the function $c^*_1(x)$. They are defined in the same way as $c^*$ and $c^*(x)$ with the only difference that, instead of arbitrary distribution functions $F_j \in \mathcal{F}^3$, we take symmetric distribution functions from the class $\mathcal{F}^3$ in the definition of the distribution function $F$.

Similarly to $\rho(l)$, we also define $\rho_*(l) = \sup \inf_{Q \in \mathcal{N}} \rho(F, Q)$, where the supremum is taken over all distribution functions $F_1, \ldots, F_n$ such that $L = l$. We will calculate the constant $c_* = \lim \sup \frac{1}{l} \rho_*(l)$. This problem arose in [4] along with other problems.

Asymptotic representation (2) is not sufficient for solving the above problems. We need a representation where the remainder term is uniformly estimated by the Lyapunov fraction. This coincides with Kolmogorov’s plan of sharpening the remainder terms in the Lyapunov theorem [1].

We present a result extending representation (2) to the case of differently distributed random variables. To formulate it, we need a series of definitions. We let $c$ denote various positive absolute constants, and let $\theta \in \mathbb{R}$ be such that $|\theta| \leq 1$. Assume that $L \leq e^{-1}$. Let $\varepsilon = L^{1+c_2} \log L$ ($c_2 = 1/55$) and let $M_1(F) = \max |\varphi(t; F)|$, where maximum is taken over $t \in I = [\frac{1}{4 \varepsilon}, \frac{2}{\varepsilon} \log |\varepsilon|]$ and $\varphi(t; F)$ is the characteristic function of the distribution function $F$. We denote by $J$ a subset of the index set, and by $F_J$ the distribution function of the sum $\frac{1}{n} \sum_{j \in J} X_j$. Then the characteristic function of $F_J$ is $\varphi(t; F_J) = \prod_{j \in J} \varphi(t; F_j)$, where $\varphi(t; F_j)$ are characteristic functions of the distribution functions $F_j$. If $J = 0$, then $F_J$ is defined as the distribution function of an identically vanishing random variable. Let $m(J) = \sigma^{-2} \sum_{j \in J} \sigma_j^2$. We call $m(J)$ the measure of the index set $J$.

Let $M_1(F) \geq L^2$. Let $J \subset \{1, 2, \ldots, n\}$ be fixed and consider the set

$$T(J) = \{ t \in I : |\varphi(t; F_J)| \geq L^2, \quad \frac{d}{dt} |\varphi(t; F)| = 0, \}$$

$$|\varphi(t; F_C J)| \geq L^{c_2} \},$$

where $CJ = \{1, 2, \ldots, n\} \setminus J$. If $T(J) \neq \emptyset$, then let $t(J)$ denote the minimal point of this set. If $T(J) = \emptyset$, then $t(J)$ denotes the minimal point $t \in I$ such that $|\varphi(t; F)| = M_1(F)$.

The quantity

$$\alpha(x) = \sum_{j=1}^n \int_{-\infty}^{\infty} \left( \Phi \left( x - \frac{u}{\sigma} \right) - \Phi(x) + \Phi'(x) \frac{u}{\sigma} - \Phi''(x) \frac{u^2}{2\sigma^2} \right) dF_j(u)$$

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