INVARIANT SUBSPACES OF A LIGHTGUIDE WITH
PERIODIC BOUNDARY

V. I. Derguzov

A waveguide operator is defined. It is proved that its spectrum coincides with the spectrum of a lightguide.
The classification of singular points of the continuous spectrum is given. Invariant subspaces of the waveguide
operator are distinguished that are related to an interval of the continuous spectrum without singular points.
Bibliography: 9 titles.

The media interface in the plane \( \mathbb{R}^2 \) of a filmlike lightguide [1, 2] is prescribed by non-self-intersecting lines \( \Gamma_1 \) and \( \Gamma_2 \) that do not meet each other and are periodic along the axis of the
lightguide, i.e., \((x, y) \in \Gamma_{1,2} \Rightarrow (x, y + 2\pi) \in \Gamma_{1,2}\). The lines \( \Gamma_1 \) and \( \Gamma_2 \) are twice continuously
differentiable. The part of the plane \( \mathbb{R}^2 \) contained between the lines \( \Gamma_1 \) and \( \Gamma_2 \) will be called a
film \( P \). We introduce the dielectric permittivity

\[
\varepsilon(x, y) = \begin{cases} 
\varepsilon_- & \text{for } (x, y) \in \mathbb{R}^2 \setminus \overline{P}, \\
\varepsilon_+(x, y) & \text{for } (x, y) \in \partial P.
\end{cases}
\]

where \( \overline{P} \) is the closure of the open set \( P \); \( \varepsilon_- \) is a positive constant; \( \varepsilon_+(x, y + 2\pi) = \varepsilon_+(x, y) \in C^1(\overline{P}) \)
denotes the class of continuously differentiable functions; and \( \varepsilon_+(x, y) > \varepsilon_- \) for \( (x, y) \in \overline{P} \). One of
the components of the electromagnetic field is a solution of the equation

\[
\left( \frac{\partial}{\partial x} \varepsilon^{-1} \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \varepsilon^{-1} \frac{\partial}{\partial y} \right) u = -\omega^2 \mu u;
\]

here \( \mu \) stands for the magnetic permeability that takes a constant positive value in \( \mathbb{R}^2 \) and \( \omega (\omega > 0) \)
denotes the frequency of oscillations. On the media interface \( \Gamma = \Gamma_1 \cup \Gamma_2 \) the function \( u \) satisfies the
following coupling condition:

\[
[u] = 0, \quad \left[ \varepsilon_+ \frac{\partial u}{\partial n} \right] = 0.
\]

In these formulas \([u] = u_+ - u_-\) and \( u_\pm \) are the limiting values of the function \( u \) on \( \Gamma \) computed from
the inside and the outside of the film \( P \); \( n \) is the outward (in relation to \( P \)) normal to the media
interface \( \Gamma \); and \( \frac{\partial u}{\partial n} \) is the derivative of the function \( u \) along the normal.

A spectral parameter \( \xi \) naturally arises when one seeks solutions of the form \( u = e^{-\xi y} v(x, y) \),
\( v(x, y + 2\pi) = v(x, y) \). In [2, 3] the spectrum \( \sigma \) of the lightguide was defined and studied. It consists
of the point spectrum \( \sigma_p \) and the continuous spectrum \( \sigma_c \) \( (\sigma = \sigma_p \cup \sigma_c) \).

In the present paper, we define a waveguide operator \( \bar{A} \) and study its spectrum \( \sigma(\bar{A}) \). The
point spectrum and the continuous spectrum of the lightguide coincide with those of the operator \( \bar{A} \)
respectively. For intervals of the continuous spectrum of general location it is proved that there exist
invariant subspaces, and the whole space can be represented as the sum of the invariant subspaces of
the operator \( \bar{A} \). We consider singular points of the continuous spectrum of a lightguide and determine
the interval of the continuous spectrum of general location from them. Similar questions for a simpler
periodic lightguide were solved earlier [4, 5].

§1. The Waveguide Operator $\overline{A}$ and Its Spectrum

Let $\Pi = \{(x, y)| x \in \mathbb{R}^1, 0 < y < 2\pi\}$ be a strip in the plane $\mathbb{R}^2$. Denote by $L^2_2(\Pi)$ the space of square-integrable functions equipped with the inner product

$$\langle F, G \rangle = \int_{\Pi} (f_1 \overline{g}_1 + f_2 \overline{g}_2) \, dx \, dy,$$

where $F = (f_1, f_2)^T$ and $G = (g_1, g_2)^T$. We introduce the domain $\Omega = \{(x, y)| (x, y) \in \Pi, (x, y) \in P\}$ and consider its part $S = \{(x, y)| (x, y) \in \Gamma \cap \Gamma_2, (x, y) \in \Pi\}$. In the space $L^2_2(\Pi)$ we define the operator

$$A(v, w)^T = \left( -\frac{\partial}{\partial x} \varepsilon \quad 0 \quad \omega^2 \mu - \varepsilon \right) \begin{pmatrix} v \\ w \end{pmatrix} + \frac{\partial}{\partial y}$$

with the domain of definition $D(A)$ constituted by $(v, w) \in L^2_2(\Pi)$ such that there exist generalized derivatives $v_x, v_y, w_y, v_{xx}$ in $\Omega$ and in $\Pi \setminus \Omega$ and $v_x, v_y, w_y, v_{xx} \in L^2(\Omega) \cap L^2(\Pi \setminus \Omega)$; the periodicity condition

$$v(x, 2\pi) = v(x, 0), \quad w(x, 2\pi) = w(x, 0)$$

and the coupling condition

$$[v] = 0, \quad [\varepsilon v_x \cos(n, x) - w \cos(n, y)] = 0$$

hold in the set $S$; here $\cos(n, x)$ and $\cos(n, y)$ denote the direction cosines of the normal $n$ to the line $S$. The periodicity condition as well as the coupling condition are understood in the sense of the imbedding theorems.

We define an anti-Hermitian operator $J$ and the indefinite inner product $\langle \langle F, G \rangle \rangle$ in the space $L^2_2(\Pi)$ by the following formulas:

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \langle \langle F, G \rangle \rangle = \frac{1}{i}(JF, G).$$

Integrating by parts, one can verify that the operator $A$ is $J$-anti-Hermitian:

$$\langle \langle AF, G \rangle \rangle = -\langle \langle F, AG \rangle \rangle$$

for $F, G \in D(A)$.

**Theorem 1.** The operator $A$ with domain of definition $D(A)$ admits the closure $\overline{A}$ in the space $L^2_2(\Pi)$. The operator $\overline{A}$ is a $J$-anti-Hermitian operator, and its spectrum $\sigma(\overline{A})$ is symmetrically located in relation to the real and imaginary axes. The spectrum $\overline{A}$ is invariant under the action of the $\iota k$-shift operator, i.e., if $\xi \in \sigma(\overline{A})$, then $\xi + \iota k \in \sigma(\overline{A})$ for any integer $k$. The spectrum, the point spectrum, and the continuous spectrum of the operator $\overline{A}$ are equal to the spectrum $\sigma$, the point spectrum $\sigma_p$, and the continuous spectrum $\sigma_c$ of the lightguide respectively: $\sigma(\overline{A}) = \sigma$, $\sigma_p(\overline{A}) = \sigma_p$, $\sigma_c(\overline{A}) = \sigma_c$.

For each eigenvalue $\xi \in \sigma_p(\overline{A})$ there is a finite-dimensional proper subspace, and for each point $\xi \in \sigma_p(\overline{A})$ or $\xi \not\in \sigma_c(\overline{A})$ there is a finite-dimensional normally splittable root subspace of the operator $\overline{A}$.

**Proof.** If $\xi \not\in \sigma$ (recall that $\sigma$ is the spectrum of the lightguide), then the system of differential equations

$$(A - \xi I)(v, w)^T = (f_1, f_2)^T$$

for any $f_1, f_2 \in L_2(\Pi), f_1(x, 2\pi) = f_1(x, 0)$, is uniquely solvable, and the solution belongs to $D(A)$ [6]. Therefore, there exists the inverse operator $(A - \xi I)^{-1}$ with a dense domain of definition.

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