ON THE HÖLDER REGULARITY FOR PARABOLIC QUASIVARIATIONAL INEQUALITIES OF IMPULSE CONTROL

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For arbitrary generalized solutions of parabolic impulse control problems, a local estimate of the Hörder norms is obtained. The boundedness of the Hörder norms of solutions subject to the Neumann boundary condition is proved. The results are established under the same hypotheses as in the classical problems. Bibliography: 4 titles.

§1. Introduction

The paper is devoted to estimating the Hölder norms of solutions to a parabolic quasivariational inequalities of impulse control. We establish local estimates for solutions and estimates of solutions satisfying the Neumann boundary condition as well. We study the quasivariational inequality corresponding to the nonlinear operator

\[ Lu = \frac{\partial}{\partial t} - \frac{\partial}{\partial x_i} a_i(x, t, u, u_x) + a(x, t, u, u_x). \]

Let \( \Omega \) be a bounded domain in \( n \)-dimensional Euclidean space \( \mathbb{R}^n \) and \( \partial \Omega \) be the boundary of the domain \( \Omega \). For a vector \( \xi = (\xi_1, \xi_2, \ldots, \xi_n) \in \mathbb{R}^n \) the notation \( \xi \geq 0 \) is equivalent to following: \( \xi_i \geq 0 \) for all \( i = 1, \ldots, n \).

We define the operator \( M \) as follows:

\[ \Psi(x, t) \equiv Mu(x, t) = 1 + \inf_{\xi \geq 0} u(x + \xi, t) \quad (1.1) \]

and introduce the sets

\[ K^0_{Mu} = \{ v \in W_2^{1,0}, v(x, t) \leq Mu(x, t) \forall t \in (0, T), \text{ a.a. } x \in \Omega \}, \]

\[ K_{Mu} = \{ v \in W_2^{1,0}, v(x, t) \leq Mu(x, t) \forall t \in (0, T), \text{ a.a. } x \in \Omega \}. \]

The definitions of spaces are taken from [1].

**Problem 1.** Find a function \( u(x, t) \in K^0_{Mu} \cap W_2^{1,1}(Q) \) such that for all \( v \in K^0_{Mu} \) the following inequality holds:

\[ \int_0^T \int_{\Omega} [u_t(v - u) + a_i(x, t, u, u_x)(v - u) x_i + a(x, t, u, u_x)(v - u)] \, dx \, dt \geq 0, \quad (1.2) \]

\[ u(0) = 0. \]

Problem 1 presents the quasivariational inequality of impulse control with the Dirichlet boundary condition.

Problem 2. Find a function \( u \in K_{Mu} \cap W_2^{1,1}(Q) \cap L_\infty(Q) \) such that for all \( v \in K_{Mu} \) the inequality (1.2) holds.

Problem 2 presents the quasivariational inequality of impulse control with the Neumann boundary condition.

In the parabolic case the question of the regularity of such solutions is complex because \( a \) \( \text{priori} \) even the one-sided smoothness of \( Mu \) in \( t \) with a fixed \( x \) is unknown. Hence it is impossible to apply the well-known results on the regularity of solutions of obstacle problems.

We now review the known results on the regularity of solutions to parabolic quasivariational inequalities of impulse control. The authors of [2–4] investigated the smoothness of solutions to quasivariational parabolic inequalities of impulse control corresponding to the operator

\[
Lu = \frac{\partial}{\partial t} - \frac{\partial}{\partial x_i}(a_{ij}(x, t) u_{x_j}) + a(x, t, u, u_x)
\]

in the case of the Dirichlet boundary condition. Biroli [2] obtained a conditional result. He proved that the Hölder norm is bounded for every continuous solution. The result was obtained under the assumption that \( a_{ij}(x, t) \in L_\infty(Q) \) and \(|a(x, t, u, p)| < c(1 + |p|)\). Vivaldi [3] established the existence of a Hölder-continuous solution provided that \( a_{ij}(x, t) \in L_\infty(Q) \) and \(|a(x, t, u, p)| < c(1 + |p|^2)\). The author [4] proved that an arbitrary generalized solution of Problem 1 is a Hölder-continuous function, assuming that the functions \( a_{ij}(x, t, u, p) \) and \( a(x, t, u, p) \) are subject to the conditions

\[
\begin{align*}
    a_{ij}(x, t, u, p) p_i & \geq \nu p^2 - \varphi_0(x, t), \\
    |a_{ij}(x, t, u, p)| & \leq \mu_0 |p| + \varphi_1(x, t), \\
    |a(x, t, u, p)| & \leq \mu_1 p^2 + \varphi_2(x, t),
\end{align*}
\]

where \( \nu \) and \( \mu_i \) are positive constants, the functions \( \varphi_i \) are nonnegative and have finite norms:

\[
\|\varphi_0, \varphi_2\|_{q, l, Q} + \|\varphi_1\|_{2q, 2l, Q} < \mu_2.
\]

In (1.4) we have used the notation

\[
\|\varphi\|_{q, l, Q} = \left( \int_0^T \left( \int_\Omega |\varphi(x, t)|^q dx \right)^{\frac{l}{q}} dt \right)^{\frac{1}{l}}.
\]

Here \( q \) and \( l \) are arbitrary positive numbers satisfying the conditions

\[
\frac{1}{l} + \frac{n}{2q} = 1 - \nu_1;
\]

moreover

\[
q \in \left[ \frac{n}{2q}, \infty \right], \quad l \in \left[ \frac{1}{1 - \nu_1}, \infty \right], \quad 0 < \nu_1 < 1 \quad \text{for} \quad n \geq 2,
\]

\[
q \in [1, \infty], \quad l \in \left[ \frac{1}{1 - \nu_1}, \frac{2}{1 - 2\nu_1} \right], \quad 0 < \nu_1 < 1/2 \quad \text{for} \quad n = 1.
\]

In the sequel, the conditions (1.3)–(1.5) are assumed to be satisfied.

In [4], the estimate of the Hölder norms of solutions is essentially based on the inequality

\[
u(x, t) \leq c\rho_0^q(x),
\]

where