AN INTERIOR ESTIMATE OF THE GRADIENT OF A SOLUTION TO THE DIRICHLET PROBLEM FOR EQUATIONS OF CURVATURE TYPE

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The Dirichlet problem for equations of curvature type is considered. It is proved that the gradient of the solution is a priori bounded at an interior point of the domain. Bibliography: 7 titles.

§1. Introduction

On $C^2(\Omega)$ we define a second-order differential operator $F(A, u)$ as follows. Let $u \in C^2(\Omega)$, and let $A(p)$ be a positive-definite symmetric matrix, $p \in \mathbb{R}^n$. Consider extremal values $k^1(A, u), \ldots, k^n(A, u)$ of the ratio of the quadratic forms:

$$
\frac{(u_{xx} dx, dx)}{(A(u_x) dx, dx)};
$$

here $u_{xx}$ denotes the Hessian matrix of the function $u(x)$, $u_i = \frac{\partial u}{\partial x_i}$. The vector of extremal values $k = (k^1, \ldots, k^n)$ can be represented as a second-order differential operator $k(A)[u]$ in $C^2(\Omega)$. Let $\mathcal{F}$ be a scalar function defined in $\mathbb{R}^n$. We introduce the operator

$$
F[A, u](x) = \mathcal{F}(k(A))[u](x), \quad x \in \Omega.
$$

Examples of such operators are operators of Monge–Ampère type, $A = E$, or the curvature operators when $A(p) = \sqrt{1 + p^2} (\partial_i^2 + p^i p^j)$. In [1-3], the classical solvability for the Dirichlet problems is established in the case $\mathcal{F}(x) = S_m(k)\left(\begin{smallmatrix} 1 \\ \vdots \\ m \end{smallmatrix}\right)$, where $S_m(k)$ is an elementary symmetric function of order $m$. In [4-6] a larger class of functions $\mathcal{F}$ is studied, for which the Dirichlet problem is solvable. In the papers just cited the matrices $A = E$ or $A(p) = \sqrt{1 + p^2} (\partial_i^2 + p^i p^j)$ were considered. In the present paper we study the curvature equations corresponding to the matrix

$$
A(p) = (a^2 + p^2)^{1-\frac{s}{2}} \left(\partial_i^2 - (s-2) \frac{p^i p^j}{a^2 + (s-1)p^2}\right), \quad s > 1, \ a \geq 0.
$$

We note that $A(p) = (v^{PP})^C$, where $v(p) = \frac{1}{2}(a^2 + p^2)^{\frac{s}{2}}$, and $v^{PP}$ stands for the Hessian matrix of the function $v(p)$. The equations with such matrices were introduced in [7], where it was proved that the Dirichlet problem is solvable for the curvature-type equations of order $m$, i.e., $\mathcal{F} = S_m\left(\begin{smallmatrix} 1 \\ \vdots \\ m \end{smallmatrix}\right)$. The present article can be regarded as a continuation of [7], which makes it possible to pass to the class of functions $\mathcal{F}$ introduced in [4-6].

§2. Statement of the Problem

We study solutions of the Dirichlet problem

$$
F[A, u] = f(x, u, u_x),
$$

$$
u_{\partial \Omega} = \Phi(x)
$$

in a bounded domain $\Omega \subset \mathbb{R}^n$. Following Trudinger [6], we define in $\mathbb{R}^n$ the numerical cone
\[ \mathcal{K}(F) = \{ k \in \mathbb{R}^n; F(k + \eta) \geq F(k) \geq 0 \} \]
\[ \forall \eta = (\eta^1, \ldots, \eta^n) \in \mathbb{R}^n, \eta^i \geq 0, i = 1, \ldots, n \} \].
Together with the numerical cone $\mathcal{K}(F)$, we consider the functional cone
\[ K(F)(x) = \{ u \in C^2(\Omega); k[u](x) \in \mathcal{K}(F)(x) \forall x \in \Omega \} \].
We assume that the function $F$ satisfies the following conditions:
\[ F \in C^1(\mathcal{K}), \]
\[ F(k + \eta) \geq F(k) \quad \forall k \in \mathcal{K}, \eta \in \mathbb{R}^n, \eta^i \geq 0, i = 1, \ldots, n, \]
\[ \sum_i F_i k^i \geq 0 \quad \text{in } \mathcal{K}, \]
\[ F_i(k) \geq \nu(f) \quad \forall k \in \mathcal{K} \text{ such that } k^i \leq 0, \]
where $\nu$ is a positive nondecreasing function defined in $\mathbb{R}^+$ and $F_i = \frac{\partial F}{\partial x^i}$.

§3. Some Algebraic Relations

To construct an a priori estimate of the gradient of a solution of Eq. (2) at interior points of the domain $\Omega$, we apply the method proposed in [7] for the function $F = S_m \setminus \binom{n}{m}$. The method is based on the classical maximum principle for elliptic equations and is connected with the differentiation of equations.

For convenient notation we pass to the other terminology.
The vector of extremal values $k$ of the ratio of the quadratic forms (1) can be regarded as the vector of eigenvalues of the problem
\[ u_{xx} \tau_\alpha = k^\alpha A \tau_\alpha, \]
\[ (A(u_x) \tau_\alpha, \tau_\beta) = \partial^\beta_\alpha, \quad \alpha, \beta = 1, \ldots, n. \]
We introduce the notation
\[ k^\alpha = u_{\alpha \alpha}, \quad u_\alpha = u_i \tau^i_\alpha, \quad u_{\alpha \eta^i} = u_i, \quad \alpha, \beta = 1, \ldots, n. \]
In view of the definition of the vectors $\tau_\alpha$, $\alpha = 1, \ldots, n$, the following equality holds:
\[ u_{\alpha \beta} = u_i \tau^i_\alpha \tau^j_\beta = \partial^\beta_\alpha u_{\alpha \alpha}. \]
On account of the specific properties of the matrix $A$, we write some equalities for the vectors $\tau_\alpha(A)$, $\eta^\alpha(A)$:
\[ \tau^i_\alpha \tau^j_\beta = (a^2 + u^2_x)^{\frac{1}{2}} \partial^\beta_\alpha (s - 2) \frac{u_i u_j}{a^2 + (s - 1) u^2_x}, \]
\[ \eta^\alpha_\eta^\beta = (a^2 + u^2_x)^{\frac{1}{2}} \partial^\beta_\alpha \left( \frac{u_i u_j}{a^2 + (s - 1) u^2_x} \right), \]
\[ \eta^\alpha_x \eta^\beta_x = (a^2 + u^2_x)^{\frac{1}{2}} \partial^\beta_\alpha \left( \frac{u_i u_j}{a^2 + (s - 1) u^2_x} \right), \]
\[ \sum_{\alpha} u^\alpha = \frac{a^2 + (s - 1) u^2_x}{(a^2 + u^2_x)^{2 - \frac{1}{2}} u^2_x}. \]