THE FUNCTIONAL CALCULUS OF FULL OPERATORS WITH DISCRETE SPECTRUM

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We construct the functional calculus for full operators with discrete spectrum over Banach spaces in the interpolation classes of symbols associated with given operators. We describe new classes of full operators in Banach spaces.

1. In a Banach space \((X, \| \cdot \|_X)\) over the field of complex numbers \(\mathbb{C}\) we consider a closed linear operator

\[ A : D(A) \subset X \to X \]

having a dense domain of definition \(D(A)\). We assume that the operator \(A\) has discrete spectrum \(\sigma(A)\). This means that the set \(\sigma(A)\) is made up of a sequence of eigenvalues \(\{\lambda_k\}_{k=1}^{\infty} \subset \mathbb{C}\) such that \(\lim_{k \to \infty} \lambda_k = \infty\) and to each eigenvalue \(\lambda_k\) there corresponds a finite-dimensional root space

\[ R(\lambda_k) = \ker (\lambda_k - A)^s, \]

where \(s\) is the order of the eigenvalue \(\lambda_k\) (that is, \(s\) is the smallest number \(s \in \mathbb{N}\) for which the equality \(\ker (\lambda_k - A)^s = \ker (\lambda_k - A)^{s+1}\) holds).

We introduce the subspace \(R = \lim\ \text{ind} \ W_n\), where \(W_n = \bigoplus_{k=1}^{\infty} R(\lambda_k)\) is the direct sum of the root subspaces and the inductive limit is taken over the embeddings \(W_n \subset W_{n+1}\), and the norm of the space \(X\) is given on the subspaces \(R(\lambda_k), W_n\). Naturally \(R \subset D(A)\) and the subspaces \(R, W_n\) are invariant relative to the operator \(A\). For that reason we can define the restrictions of the operator \(A\) to them, which we denote by \(A_R\) and \(A_n\) respectively.

With any eigenvalue \(\lambda_k\) we associate an \(r_k\)-dimensional complex vector space

\[ C_{\lambda_k} = \{ f(\lambda_k) : f(\lambda_k) = [f(0)(\lambda_k), \ldots, f(r_k-1)(\lambda_k)] \} \]

with the norm

\[ \|f(\lambda_k)\| = \left\| f(0)(\lambda_k) \right\| P(\lambda_k) + \sum_{j=1}^{r_k-1} \frac{\| f(j)(\lambda_k) \|}{j!} \left\| (\lambda_k - A)^j P(\lambda_k) \right\|, \]

where \(P(\lambda_k) = \frac{1}{2\pi i} \int (\lambda - A)^{-1} d\lambda\) is the Riesz projection on the root subspace \(R(\lambda_k)\) and the norms of all operators are defined in the space \(X\). It is not difficult to verify by direct computation that the following proposition holds.

Lemma 1. The space \(C_{\lambda_k}^{(r_k)}\) is a commutative Banach algebra with respect to the convolution

\[ f(\lambda_k) * g(\lambda_k) = [h^{(0)}(\lambda_k), \ldots, h^{(r_k-1)}(\lambda_k)], \]

where

\[ h^{(m)}(\lambda_k) = \sum_{j=0}^{m} C_m^j f^{(j)}(\lambda_k) g^{(m-j)}(\lambda_k), \quad C_m^j = \frac{m!}{j! (m-j)!}, \]

\[ g(\lambda_k) = [g^{(0)}(\lambda_k), \ldots, g^{(r_k-1)}(\lambda_k)], \quad m = 0, r_k - 1, \]

and has an identity \(e(\lambda_k) = (1, 0, \ldots, 0)\). The spectrum of the algebra \(C_{\lambda_k}^{(r_k)}\) is made up of the single character \(\hat{\lambda}_k : f(\lambda_k) \to f(0)(\lambda_k)\) and the maximal ideal corresponding to it coincides with the radical \(\text{Rad} \ C_{\lambda_k}^{(r_k)} = \{ f(\lambda_k) : f(0)(\lambda_k) = 0 \}\) and has dimension \(r_k - 1\).

With the subspace \(W_n\) we associate the direct product of algebras
We now construct the projective limit of the Banach algebras \( C^{(r_1,...,r_n)}_{\lambda_1,...,\lambda_n} \): 
\[
C^{(r_1,...,r_n)}_{\sigma(A)} = \lim_{n \to \infty} \text{pr} C^{(r_1,...,r_n)}_{\lambda_1,...,\lambda_n},
\]
where \( C^{(r_1,...,r_n)}_{\sigma(A)} \ni f \to f_n \in C^{(r_1,...,r_n)}_{\lambda_1,...,\lambda_n} \) are the canonical projections and \( f = \{ f_n \}_{n=1}^{\infty} \).

**Lemma 2.** The projective limit \( C^{(r)}_{\sigma(A)} \) is a commutative Fréchet algebra with respect to the multiplication \( f \circ g = \{ f_n \circ g_n \} \) and \( e = \{ e_n \} \) is its identity. The topology of \( C^{(r)}_{\sigma(A)} \) is generated by the sequence of pseudonorms \( d_n(f) = \| f_n \| \), which satisfy the relations 
\[
d_n(f) \leq d_{n+1}(f), \quad d_n(f \circ g) \leq d_n(f) d_n(g), \quad d_n(e) = 1,
\]
for all \( n \in \mathbb{N} \) and \( f, g \in C^{(r)}_{\sigma(A)} \).

The topological spectrum of the algebra \( C^{(r)}_{\sigma(A)} \) (the set of continuous characters with the weak topology, induced by the conjugate space to \( C^{(r)}_{\sigma(A)} \)) coincides with the operator spectrum \( \sigma(A) \), endowed with the discrete topology, and the corresponding Gelfand transform 
\[
\sigma^{(r)}_{\sigma(A)} \ni f \mapsto \hat{f} \equiv \{ f^{(0)}(\lambda_k) \} \in \prod_{k=1}^{n} \mathbb{C}
\]
establishes a topological homomorphism of the algebra \( C^{(r)}_{\sigma(A)} \) onto the countable direct product of the field \( \mathbb{C} \) with coordinatewise multiplication of elements. In particular, the spectrum \( \sigma(f) \) of an arbitrary element \( f \) in the algebra \( C^{(r)}_{\sigma(A)} \) coincides with the sequence of numbers \( \{ f^{(0)}(\lambda_k) \}_{k=1}^{\infty} \).

**Proof.** In accordance with Lemma 1 the spectrum \( \hat{\lambda}_k \) of each of the algebras \( C^{(r)}_{\lambda_k} \) can be identified with the eigenvalue \( \lambda_k \in \mathbb{C} \). Therefore the spectrum of their direct product \( C^{(r_1,...,r_n)}_{\lambda_1,...,\lambda_n} \) is the set of eigenvalues \( \{ \lambda_1,...,\lambda_n \} \). To compute the topological spectrum of the projective limit \( \lim_{n \to \infty} \text{pr} C^{(r_1,...,r_n)}_{\lambda_1,...,\lambda_n} \) we apply the known duality relations for locally convex algebras [4]. According to these relations the topological spectrum equals the inductive limit 
\[
\lim_{n \to \infty} \text{ind} \{ \lambda_1,...,\lambda_n \} = \bigcup_{n=1}^{\infty} \{ \lambda_1,...,\lambda_n \}
\]
with respect to the embeddings \( \{ \lambda_1,...,\lambda_n \} \subset \{ \lambda_1,...,\lambda_{n+1} \} \). That is to say, 
\[
\lim_{n \to \infty} \text{ind} \{ \lambda_1,...,\lambda_n \} = \sigma(A) \text{ in the discrete topology of the set } \sigma(A).
\]

For any element \( f \in C^{(r)}_{\sigma(A)} \) the relation 
\[
\sigma(f) = \{ h(f) : h \in M(C^{(r)}_{\sigma(A)}) \},
\]
holds, where \( \sigma(f) \) is the spectrum of the element \( f \) in the algebra \( C^{(r)}_{\sigma(A)} \); \( M(C^{(r)}_{\sigma(A)}) \) is the topological spectrum of the algebra \( C^{(r)}_{\sigma(A)} \) [4]. Hence in particular it follows that the spectrum of any element \( f \) in \( C^{(r)}_{\sigma(A)} \) can be computed using its Gelfand transform over the topological spectrum, that is, 
\[
\sigma(f) = \{ f^{(0)}(\lambda_k) \}_{k=1}^{\infty}.
\]

Finally, the fact that the transformation \( f \mapsto \hat{f} \) gives a topological homomorphism follows from its surjectiveness and the open mapping theorem. The lemma is now proved.

In the finite-dimensional algebra \( L(W_n) \) of linear operators on the space \( W_n \) we consider the subalgebra \( L_0(W_n) \)