ON AN OPERATIONAL METHOD OF SOLVING INITIAL-VALUE PROBLEMS FOR PARTIAL DIFFERENTIAL EQUATIONS INDUCED BY GENERALIZED SEPARATION OF VARIABLES

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We propose an operational method of solving the Cauchy problem for partial differential equations and systems of partial differential equations. We demonstrate its superiority to the known methods. We give a number of illustrative examples of applications of the method.

The efforts of many scholars have amassed a large number of methods of solving both partial differential equations and initial-value problems related to them. Among such methods are the well-known operator method [1, 7, 16], the method of integral transforms [22], the method of initial functions [6] and others. The application of each method to a specific Cauchy problem, wherever this can be done, makes it possible to obtain different presentations of its solution. To the methods of solving the Cauchy problem mentioned above one may add the operational method [9–11, 13, 19], which is based on generalized separation of variables [9, 12, 14]. This method is in a certain sense "dual" to the operator method whose starting point is the classical operational calculus.

The classical operational calculus is a method of solving ordinary differential equations in which the symbol 
\[ p = \frac{d}{dx} \]

is manipulated like an ordinary variable. For the required solution one obtains expressions of the form
\[ g(p)f(x), \]

where \( f(x) \) is a known function. Such a calculus was first proposed in [4], and then greatly developed in [25]. The basic problem in this calculus is obviously to find an effective interpretation of the symbols \( g(p) \). We note that a complete mathematical justification of the symbolic calculus was not given in these papers. The justification was given much later, when a connection was established between this calculus and the Laplace transform. What is important in this topic was the algebraic approach to justifying the operational calculus proposed in [17].

The technique of infinite-order differential operators which is inherent in the classical symbolic calculus, has recently been successfully applied to study Cauchy problems for partial differential equations (see, for example, [7, 8] and the bibliography they contain).

The formalism of infinite-order differential operators has frequently been applied in recent years to study specific problems of mechanics [3, 6, 15, 20]. In these papers the solutions of initial- and boundary-value problems are given using the expressions (1) where \( x \in \mathbb{R}^n, \ p = (p_1, p_2, \ldots, p_n), \ p_j = \partial / \partial x_j, \ j = 1, n. \)

The operational method of generalized separation of variables considered in the present paper makes it possible to solve the Cauchy problem using expressions of the form

\[ f_j \left( \frac{\partial}{\partial \mu} \right) \left[ g_j(x, \mu) \right]_{\mu=0}^\# \]

where \( f_j \) are known functions (initial functions or right-hand sides of the equations), and \( \mu \in \mathbb{C}^n. \)

The operational method of generalized separation of variables has several advantages over those mentioned above, and the present paper is devoted to a description of those advantages. The characteristics of the method can be illustrated using the examples of Cauchy problems for the equations of the direct and inverse heat-conduction problems, wave equations for the cases of two and three spatial variables, and the Lamé equations.


1.1. The heat equation

\[ \left( \frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} \right) u(t, x) = 0, \quad u(0, x) = \varphi(x), \quad t > 0, \ x \in \mathbb{R}^1. \]
The application of the different methods of solving the problem (2) makes it possible to write the following formal solutions of it:

\[ u(t, x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left[-t\xi^2 + i\xi x\right] \hat{\phi}(\xi) \, d\xi, \]  

(3)

\[ u(t, x) = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} \phi(\xi) \exp\left[-\frac{(x - \xi)^2}{4t}\right] d\xi, \]  

(4)

\[ u(t, x) = \exp\left[ t \frac{d^2}{dx^2}\right] \varphi(x). \]  

(5)

Formula (3) for the solution of the problem (2) is obtained using the Fourier transform. Here \(\hat{\phi}(\xi)\) is the Fourier transform of the initial function \(\varphi(x)\):

\[ \hat{\varphi}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \varphi(x) \exp\left[-ix\xi\right] dx. \]

There exists a unique solution (3) of the problem (2), for example, in the space of functions \(u(t, x)\) that belong to \(H^\infty(\mathbb{R}^4)\) for each \(t \in \mathbb{R}_+\), and are continuous functions of \(t\) together with their first time derivative, where [7]

\[ H^\infty(\mathbb{R}^4) = \left\{ \varphi(x) \in L_2(\mathbb{R}^4): \hat{\varphi}(\xi) \text{ is of compact support in } \mathbb{R}^4 \right\}. \]

Formula (4) for solving the problem (2) is called Poisson's formula [5]. It is proved in [23] that the class in which a unique solution (4) of the problem (2) exists is the class of functions that for each \(t \geq 0\) grow more slowly than \(\exp[x^2]\) as \(|x| \to \infty\).

In order for a solution (5) of the problem (2) to exist the function \(\varphi\) must belong to the Sobolev space of infinite order [7]

\[ \mathcal{W}^\infty = \left\{ \varphi(x): \sum_{n=0}^{\infty} \frac{t^n}{n!} \left\| \varphi^{(2n)}(x) \right\|_{L_2(\mathbb{R}^4)} < \infty \right\}. \]

Applying the operational method of generalized separation of variables to the problem (2) makes it possible to obtain the following formal solution [9]:

\[ u(t, x) = \varphi\left( \frac{\partial}{\partial t} \right) \{ \exp[\mu^2 t + \mu x] \}_{\mu=0}. \]  

(6)

The existence and uniqueness class for the solution (6) is the class of functions that have a single-valued analytic continuation to entire functions of order less than 2 for each \(t \geq 0\) [9].

Formulas (3)–(6), as we see, are applicable for different classes of initial functions. In particular, if \(\varphi(x)\) is a polynomial, formula (3) is inapplicable, and formula (4), though it does apply, involves computing improper Euler integrals of second kind. Thus, if \(\varphi(x) = 1\), then, taking account of the relation \(\int_{-\infty}^{\infty} \exp\left[-\frac{(x - \xi)^2}{4t}\right] d\xi = 2\sqrt{\pi} t\), we find

\[ u(t, x) = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} \exp\left[-\frac{(x - \xi)^2}{4t}\right] d\xi = \frac{1}{2\sqrt{\pi t}} \cdot 2\sqrt{\pi} t = 1. \]

This result can be achieved much more rapidly with formulas (5) and (6), and moreover without computing the Euler integral. Setting \(\varphi(x) = 1\) in those formulas, we obtain

\[ u(t, x) = \exp\left[ t \frac{d^2}{dx^2}\right] \cdot 1 = \left( 1 + t \frac{d^2}{dx^2} + \frac{1}{2!} t^2 \frac{d^4}{dx^4} + \ldots \right) \cdot 1 = 1, \]

\[ u(t, x) = 1 \left\{ \exp[\mu^2 t + \mu x] \right\}_{\mu=0} = \exp[0^2 t + 0 x] = 1. \]