THE R-FUNCTION METHOD IN BOUNDARY-VALUE PROBLEMS WITH GEOMETRIC AND PHYSICAL SYMMETRY

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This paper is devoted to developing constructive methods of the theory of R-functions. It gives the first discussion of methods of constructing the equations of loci with symmetry of translation type for duplicated domains not separated by lines and with point symmetry of cyclic type for regions both possessing and lacking axial symmetry. The possible applications are illustrated by numerous examples carried out using the POLE system, including solutions of boundary-value problems involving the influence of cyclically located circular and star-shaped insulators on an electric field.

The traditional mathematical models of fields of various physical nature are boundary-value problems for partial differential equations. The usual statement of a boundary-value problem has the following form:

\[ Au = f \quad \text{in the region } \Omega, \]
\[ L_i u = \varphi_i \quad \text{on } \partial \Omega_i, \quad i = 1, 2, ..., m, \]

where \( \Omega \) is the region in which the solution \( u \) is being sought; \( \partial \Omega_i, \quad i = 1, 2, ..., m, \) is a covering of the boundary \( \partial \Omega \) of the region \( \Omega \) (the parts \( \partial \Omega_i \) are not necessarily different and may be equal to \( \partial \Omega \)); \( f \) and \( \varphi_i \) are known functions (perturbing the field), vector-valued functions, tensors, or possibly elements of some other type. In the R-function method developed in the present article geometric information is taken into account by functions \( \omega, \omega_i \), chosen as a rule from among the elementary functions, so that \( \partial \Omega \) and \( \partial \Omega_i \) are the sets of their zeros, and the solution \( u \) is sought in the pencil

\[ u = B(\Phi, \omega, \omega_i) + \varphi_0, \]

Here \( \varphi_0 \) is a known function; \( B \) is an operator depending on the shape of the boundary \( \partial \Omega \) and its parts \( \partial \Omega_i, \quad i = 1, 2, ..., m, \) formed so that for any choice of an undetermined component \( \Phi \) from a set \( M \) formula (3) satisfies the boundary conditions (2) exactly. If this formula has the property of completeness relative to the boundary-value problem (1)-(2), that is, it is possible to choose an undetermined component \( \Phi \) that converts (3) into a solution of the problem with any prescribed precision (in some sense), it is called a general structure solution of the problem in question.

We recall first how the R-function method solves the problem of constructing the functions \( \omega \) and \( \omega_i \) that occur in formulas of the form (3). A complicated locus (also called a geometric object or a drawing) which the region \( \Omega \), and hence also its boundary \( \partial \Omega \), may be, can be described by a logical formula connecting simple loci (primitives) after which transition to the usual equations (resp. inequalities) of the form \( \omega(x, y) = 0 \) (resp. \( \omega(x, y) \geq 0 \)) recognized in analytic geometry is accomplished using R-functions. The technique of constructing them will be shown through examples below.

In the majority of cases the following R-functions are used to construct the functions \( \omega, \omega_i \):

\[ x \land \alpha y = \frac{1}{1+\alpha} \left( x + y - \sqrt{x^2 + y^2 - 2\alpha xy} \right), \]
\[ x \lor \alpha y = \frac{1}{1+\alpha} \left( x + y + \sqrt{x^2 + y^2 - 2\alpha xy} \right), \]
\[ \overline{x} = -x, \]
\[ x \Delta y = \frac{xy}{x+y}, \]
\[ x \land y = \frac{xy}{\sqrt{\alpha x^2 + y^2}}, \text{ where } \alpha = \alpha(x,y), \quad -1 < \alpha \leq 1. \]


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The commonest choices are $\alpha = 0, 1, 0.9$ or $\alpha = (1 + x^2 + y^2)^{-1}$ [2–5]. When necessary, the following normalization condition [2] can be implemented:

$$\frac{\partial \omega}{\partial n}|_{\omega=0} = 1. \tag{5}$$

The first two of formulas (4) correspond to the known logical operations of intersection and union of sets, the third corresponds to complementation (negation), and the other two to equivalence (negation of the symmetric difference) [2, 6]. Information on other $R$-functions, methods of constructing them, and their use, can be found in [2, 5].

Example 1. The rectangle $\Omega$ shown in Fig. 1a (we assume its width to be $2a$, and height $2b$; the other notation in this figure will be discussed below) is the intersection of two infinite strips:

$$\Omega = \Sigma_1 \cap \Sigma_2 = \left[ \sigma_1 = \frac{(a^2 - x^2)}{2a} > 0 \right] \cap \left[ \sigma_2 = \frac{(b^2 - y^2)}{2b} > 0 \right], \tag{6}$$

where the symbol "\( \cap \)" denotes the operation of set intersection. It is not difficult to verify that the functions $\sigma_1$ and $\sigma_2$, because of the factors $1/(2a)$ and $1/(2b)$, are normalized, that is, they satisfy the following conditions on the boundary of the regions corresponding to them (a strip in the present case):

$$\frac{\partial \sigma_i}{\partial n}|_{\sigma=0} = 1, \quad i = 1, 2. \tag{7}$$

According to the $R$-function method, to obtain the normalized equation of the boundary of the region defined by the formula (6), it suffices to eliminate the symbols "$\geq 0"$ in (6) and replace the operation of intersection "$\cap"$ by the symbol "$\lor"$ [2, 5]:

$$\omega = \left[ (a^2 - x^2)/(2a) \lor (b^2 - y^2)/(2b) \right] = 0. \tag{8}$$

It is not difficult to note that the left-hand side of Eq. (8), denoted by $\omega$, is an ordinary elementary function, since the symbol for the operation "$\lor"$ can be eliminated using the first formula of (4). One can proceed similarly if several regions are involved in the formation of the region $\Omega$ (examples will be given below). If the functions $\omega_i$ corresponding to the parts of the boundary must be used in the general structure solution one can go about constructing them as follows. Let $\omega = 0$ be the equation of the boundary $\partial \Omega$, and $\Sigma_i = (\sigma_i \geq 0)$ a region that distinguishes a part $\partial \Omega_i$ of the boundary $\partial \Omega$. Then if the function $\omega$ is normalized on $\partial \Omega$, the equation of the distinguished part of the boundary can be represented as

$$\omega_i = \sqrt{\omega \lor \sigma_i} = 0, \tag{9}$$

and the function $\omega_i$ will be normalized on $\partial \Omega_i = (\omega_i = 0)$.

Example 2. We write the equation $\omega_i = 0$ of the right half EABF of the boundary of the rectangle ABCD for which Eq. (8) was formed. In this case $\Sigma_3 = (\sigma_3 = x \geq 0)$. Consequently, according to (9) we obtain

$$\omega_i = \sqrt{\left[ (a^2 - x^2)/(2a) \lor (b^2 - y^2)/(2b) \right] \lor_x (-x)} = 0. \tag{10}$$

To obtain the equation $\omega_2 = 0$ of the left half FCDE of this rectangle, it suffices to change the sign of $x$: