A GENERALIZED GREEN'S FORMULA FOR ELLIPTIC PROBLEMS IN DOMAINS WITH EDGES

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The usual Green’s formula connected with the operator of a boundary-value problem fails when both of the solutions \( u \) and \( v \) that occur in it have singularities that are too strong at a conic point or at an edge on the boundary of the domain. We deduce a generalized Green’s formula that acquires an additional bilinear form in \( u \) and \( v \) and is determined by the coefficients in the expansion of solutions near singularities of the boundary. We obtain improved asymptotic representations of solutions in a neighborhood of an edge of positive dimension, which together with the generalized Green’s formula makes it possible, for example, to describe the infinite-dimensional kernel of the operator of an elliptic problem in a domain with edge.

Bibliography: 14 titles.

In elliptic problems there exist solutions that are infinitely large near singularities of the boundary (conic points, edges) even when the right-hand sides are zero in a neighborhood of the singularity. If these increasing functions are “substituted” for the pair of functions \( u \) and \( v \) in the usual Green’s formula connected with the operator of the problem, the formula acquires an additional term, a bilinear form in \( u \) and \( v \) determined by the asymptotics of \( u \) and \( v \) near singularities. We shall call such Green’s formulas with extra terms generalized Green’s formulas. They turn out to be useful in describing the domains of definition of coupled problems, self-adjoint extensions of symmetric operators of boundary problems, and other questions.

The derivation of the Green’s formulas is based on rather precise asymptotics of the solutions near edges. The necessary information on the asymptotics is contained in §§ 1 and 2. In this situation the requirements on the initial data are weakened in comparison with those in [1]. Problems in domains with conic points are discussed in § 3, and the contribution of the singularities to the Green’s formula for domains with edges is studied in § 4.

Applications are not discussed in this paper, except for the description of the cokernel of the operator of the boundary-value problem in the last section.

§ 1. Preliminary Information

1. Statement of the problem. Let \( G \) be a domain in \( \mathbb{R}^n \) with compact closure \( \overline{G} \) and \((n-1)\)-dimensional boundary \( \partial G \). Suppose that a subset \( \mathcal{M} \) is distinguished on \( \partial G \) such that:

1) \( \mathcal{M} \) is smooth (\( C^\infty \)) compact \( d \)-dimensional submanifold of \( \mathbb{R}^n \) without boundary;

2) \( \partial G \setminus \mathcal{M} \) is a smooth (noncompact) submanifold of \( \mathbb{R}^n \);

3) for each point \( x^0 \in \mathcal{M} \) there exist a neighborhood \( U \) in \( \mathbb{R}^n \) and a diffeomorphism \( \varphi : U \to \mathbb{R}^n \) subject to the conditions \( \varphi(x^0) = 0, \varphi'(x^0) = 1 \) and \( \varphi(U \cap G) = B^+_\rho(0) \cap D(x^0) \), where \( B^+_\rho(0) \) is the ball \( \{z \in \mathbb{R}^n : |z| < \rho\} \); \( D(x^0) \) is the wedge \( K(x^0) \times \mathbb{R}^d \), and \( K(x^0) \) is the open \((n-d)\)-dimensional cone with vertex \( 0 \) cutting off a domain \( \Omega(x^0) \) on the unit sphere \( S^{n-1} \) with smooth boundary \( \partial \Omega(x^0) \).

The scalar differential operator

\[
P(x, D_z) = \sum_{|\gamma| \leq q} p_\gamma(x)D_z^\gamma
\]

with coefficients in \( C^\infty(\overline{G} \setminus \mathcal{M}) \) is said to be \( \delta \)-admissible if the following representations hold in a neighborhood of each point \( x^0 \) of the edge of \( \mathcal{M} \):

\[
p_\gamma(x) = r^{d-|\gamma|}p_\gamma^0(\omega, z) + r^{2d-|\gamma|+d}\tilde{p}_\gamma(\omega, z),
\]

\[
p_\gamma^0 \in C^\infty(\overline{\Omega} \times \mathbb{R}^d), \quad (rD_r)^jD_\omega^\alpha D_z^\beta \tilde{p}_\gamma \in L_\infty(\mathbb{R}_+ \times \Omega \times \mathbb{R}^d).
\]

In these formulas $\delta > 0$, $\kappa(z) = (y, z)$, $z \in \mathbb{R}^d$, $y \in \mathbb{R}^{n-d}$, and $(r, \omega)$ are the spherical coordinates of the point $y$, $j = 0, 1, \ldots$, and $\alpha$ and $\sigma$ are arbitrary multi-indices. (In the notation $\Omega(x^0)$ and $D(x^0)$ and the like we shall omit the point $x^0$ as a rule.) We define the principal part of the operator (1.1) at the point $x^0 \in \mathcal{M}$ to be the differential expression

$$P^0(y, D_y, D_z) = \sum_{|\gamma| \leq q} r^{1 - |\gamma|} p^0_\gamma(\omega, 0) D^\gamma_{(y, z)}, \quad (1.3)$$

where $p^0_\gamma$ are the functions of (1.2).

Let $L(x, D_x)$ and $B(x, D_x)$ be a $k \times k$ matrix and an $m \times k$ matrix whose entries are $\delta$-admissible differential operators, with $\operatorname{ord} L_{hj} = s_h + t_j$, and $\operatorname{ord} B_{qj} = \sigma_q + t_j$. Assume that the boundary-value problem

$$L(x, D_x)u(x) = f(x), \quad x \in G,$$

$$B(x, D_x)u(x) = g(x), \quad x \in \partial G \setminus \mathcal{M}, \quad (1.4)$$

is (Douglis-Nirenberg) elliptic on $\overline{G} \setminus \mathcal{M}$. In addition, we introduce the condition that it be elliptic on the edge $\mathcal{M}$: for each point $x^0 \in \mathcal{M}$ the principal part $\{L^0, B^0\}$ of the operator $\{L, B\}$ is an elliptic problem in the wedge $\overline{D} \setminus \mathcal{M}$.

We define the space $V^\beta_1(G; \mathcal{M})$ as the completion of the set $C^\infty(\overline{G} \setminus \mathcal{M})$ of smooth functions with compact support in the norm:

$$\|u; V^\beta_1(G; \mathcal{M})\| = \left( \sum_{|\gamma| \leq l} \|\rho^\beta - |\gamma| D^\gamma u; L_2(G)\|^2 \right)^{1/2} \quad (1.5)$$

where $l = 0, 1, \ldots, \beta \in C^\infty(\mathcal{M})$, $\rho^{\beta-j}$ is a smooth function that is positive in $\overline{G} \setminus \mathcal{M}$ equivalent to the function $(y, z) \mapsto |y|^\beta - |z| - j$ in each neighborhood $\mathcal{U}$. Further suppose $V^{\beta-1/2} (\partial G; \mathcal{M})$ is the space of traces on $\partial G \setminus \mathcal{M}$ of functions of $V^\beta_1(G; \mathcal{M})$. It is clear that the mapping

$$A = \{L, B\}: \mathcal{D}^T_\beta V(G, \mathcal{M}) \to \mathcal{D}^T_\beta V(G, \mathcal{M}) = \prod_{h=1}^k V^\beta_{s_h}(G; \mathcal{M}) \times \prod_{q=1}^m V^\beta_{\sigma_q - 1/2} (\partial G; \mathcal{M}) \quad (1.6)$$

is continuous for $l \geq \max\{-t_j, s_h, \sigma_q + 1\}$.

2. Model problems in a wedge and in a cone. Estimating solutions of the problem (1.4) near an edge reduces to the study of the following “model” problem in the wedge:

$$L^0(y, D_y, D_z)u(y, z) = f(y, z), \quad (y, z) \in D,$$

$$B^0(y, D_y, D_z)u(y, z) = g(y, z), \quad (y, z) \in D \setminus \Gamma, \quad (1.7)$$

where $\gamma = 0 \times \mathbb{R}^d$ is an edge of the wedge $D$. The operator of this problem is realized by the continuous mapping

$$A^0 = \{L^0, B^0\}: \mathcal{D}^T_\beta V(D, \Gamma) \to \mathcal{D}^T_\beta V(D, \Gamma). \quad (1.8)$$

Here $\beta$ is a real number and the functional spaces are defined in analogy with (1.6) and (1.5) with $G, \mathcal{M}$, and $\rho$ replaced by $D, \Gamma$, and $|y|$.

We now apply the Fourier transform $F_{\mathcal{M} - \xi}$ to the problem (1.7) and introduce the notation

$$\eta = |\xi| y, \quad \theta = |\xi|^{-1} \xi, \quad U_j(\eta, \xi) = |\xi|^{1 - a_j} \tilde{u}_j(y, \xi), \quad F_k(\eta, \xi) = |\xi|^{-a} \tilde{f}_k(y, \xi), \quad G_q(\eta, \xi) = |\xi|^{-a_q} \tilde{g}_q(y, \xi),$$

where $\tilde{w}(y, \cdot)$ is the Fourier image of the function $z \mapsto w(y, z)$. As a result from (1.7) we obtain the following model problem in the cone

$$L^0(\eta, D_\eta, \theta)U(\eta, \xi) = F(\eta, \xi), \quad \eta \in K,$$

$$B^0(\eta, D_\eta, \theta)U(\eta, \xi) = G(\eta, \xi), \quad \eta \in \partial K \setminus 0. \quad (1.9)$$