The lifetime of a large repairable system can be decomposed into two parts. The first one is the cumulative time spent in the perfect state (all components are operating), and the second one is the restoration time when some components have failed. For highly reliable systems, the first time is close to the system lifetime, and it turns out that this approximation is accurate in many practical cases. Nevertheless, it is important to evaluate the error of such an approximation. Some bounds exist if each component has a constant failure rate. In this paper, using estimates of geometric sums, we get bounds for the general case.

1. Introduction

When studying a large and highly reliable system, one can notice that the cumulative time spent by the system in the perfect state is very close to its lifetime. During the lifetime, many failures of components can occur, but, owing to the fact that repair times are small, most of the system lifetime is spent in the state where all components are operating. Some classical approximations of the reliability which are used with efficiency are based on this observation. Such approximations (which are obviously pessimistic) often employ the regenerative structure of the process, where lifetime can be represented as a geometric sum of independent identically distributed random variables (i.i.d.r.v.'s). Their practical accuracy has been confirmed by theoretical results in the specific case where all components have a constant failure rate and, therefore, successive sojourn times and the cumulative sojourn-time in the perfect state are exponentially distributed (see, for example, [3, 11]).

We place our study in the same framework and propose new bounds to evaluate the accuracy of the approximation in the general case. According to our knowledge, these estimates are better than those we found in the literature.

The paper is organized as follows. In Sec. 2, we state the model and provide some classical results. In Sec. 3, we give explicit formulas for the accuracy which are used in the sequel. Section 4 contains auxiliary results, which are concerned with the sub-additivity and aging properties of distribution functions (d.f.'s) and which reveal useful properties of the desired reliability characteristics. In Sec. 5, we derive a variety of accuracy estimates and compare them with a few known results. Section 6 contains exponential bounds of the unknown reliability function. All the reasoning essentially uses the technique developed in [7].

2. Notation and Classical Results

Let us state the problem. For this, it is not necessary to describe a concrete model or give specific details concerning reliability theory. All we need is the decomposition of the lifetime appearing in numerous concrete situations (see, for example, [5]). Let the dynamics of the system until its failure be viewed as consisting of successive independent cycles, the number of cycles being a geometric r.v. $\nu$ with the distribution

$$P(\nu = k) = qp^{k-1}, \quad q = 1 - p.$$ 

In practice, $q$ represents the probability of having an unsuccessful restoration (during which the system fails). Let the length of the $i$th cycle ($1 \leq i < \nu$) be the sum of two independent r.v.'s $X_i$ and $Y_i$, which can be regarded as the time spent in the perfect and repair states, respectively, on the condition that the repair is successful (during which the system was completely restored). The length of the last cycle (having the number $\nu$) is the sum of two independent r.v.'s $X_\nu$ and $Z$, the latter representing the conditional repair time until failure given that the repair is unsuccessful. Let $\{X_i\}$, $\{Y_i\}$, $Z$, and $\nu$ be independent and, additionally, each of the sequences $\{X_i\}$ and $\{Y_i\}$ consist of i.i.d.r.v.'s.
Let $r$ denote the lifetime of a system. Then, in accordance with our assumptions,

$$
    r = \sum_{i=1}^{\nu} X_i + \sum_{i=1}^{\nu-1} Y_i + Z. \quad (2.1)
$$

We are interested in the unreliability function

$$
    W(z) = P(r \leq z), \quad (2.2)
$$

and we want to compare it with the pessimistic approximation

$$
    W_0(z) = P\left( \sum_{i=1}^{\nu} X_i \leq z \right), \quad (2.3)
$$

which takes into account only the time spent in the perfect state.

This comparison problem often appears in reliability, queueing theory, risk theory, and many other applied areas (cf. [4-6]). Indeed, one can write simple analytical expressions for $W$ and $W_0$:

$$
    W(z) = \sum_{k=1}^{\infty} q^k F_X^k * F_Y^{k-1} * F_Z, \quad (2.4)
$$

$$
    W_0(z) = \sum_{k=1}^{\infty} q^k F_X^k \quad (2.5)
$$

with obvious notations concerning the d.f.'s $F_X$, $F_Y$, and $F_Z$ of r.v.'s $X$, $Y$, and $Z$, respectively; the symbol $*$ stands for the convolution and, say, $F_X^k$ denotes the $k$-fold convolution of $F_X$.

Our goal is to obtain accurate bounds for the difference

$$
    \delta(z) = W_0(z) - W(z).
$$

The special case $F_X(z) = 1 - \exp(-\lambda z)$ was the subject of many works in the last two decades. Let us give two classical bounds. The first one is a very simple uniform bound taken from [11]:

$$
    \delta(z) \leq \lambda q E(Z) + \lambda p E(Y). \quad (2.6)
$$

The second bound is nonuniform. It refines the estimate (2.6) and can be found in [3]:

$$
    \delta(z) \leq \lambda q E(Z) + \lambda p(1 - e^{-\lambda z}) E(Y). \quad (2.7)
$$

Such bounds are particularly efficient when the probability $q$ of unsuccessful repairing is small or when the mean times $EY$ and $EZ$ are very small as compared to $E(X)$. In bounding $\delta(z)$, one can also use the variety of results established for regenerative processes (see [5, 7]) and different specific properties of $F_X$. For example, if $F_X$ has some aging properties, one can use the accuracy bound (cf. [9])

$$
    \delta(z) \leq q \frac{E(Z)}{E(X)} + p \frac{E(Y)}{E(X)} + q. \quad (2.8)
$$

When obtaining further results we shall not assume that $F_X$ is exponential and we shall try to avoid any other restrictive conditions.

Note also that all the bounds will have the form

$$
    \delta(z) \leq \bar{\delta}(z),
$$

where $\bar{\delta}(z)$ is a nondecreasing function of $z$. Since $\delta(z) \leq 1 - W(z)$, one can employ a more accurate bound

$$
    \delta(z) \leq \min(\bar{\delta}(z), 1 - W(z)),
$$

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