EXACT TOPOLOGICAL CLASSIFICATION OF HAMILTONIAN FLOWS ON SMOOTH TWO-DIMENSIONAL SURFACES

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The present paper contains an exact topological classification of all nondegenerate Hamiltonian systems on smooth closed two-dimensional surfaces. Bibliography: 8 titles.

§1. INTRODUCTION

The present paper continues the series of papers by the authors [1–3] devoted to a new theory of the topological orbital classification for integrable Hamiltonian systems of differential equations with two degrees of freedom on three-dimensional isoenergy submanifolds. It turns out that there exist quite visual and simple invariants serving to obtain a complete classification of such systems. It has also been discovered that first it is necessary to solve the simpler problem whose formulation is the title of this paper. However, this separate problem (for systems with one degree of freedom which are always integrable) is extremely interesting by itself.

The main result of the paper is stated in Theorems 1 and 2, which give an exact classification for all nondegenerate Hamiltonian systems on smooth closed two-dimensional surfaces.

To begin with, we introduce some necessary notion. In the sequel, we essentially use some ideas and terminology from [4–8].

Consider a smooth compact closed (i.e., without boundary) oriented 2-surface $P^2$, and denote by $F$ a smooth Morse function on $P^2$. Let $P^2$ be endowed with a smooth symplectic structure $\omega$. Consider the Hamiltonian vector field $w = \text{grad} F$ on $P$. In the present paper, we give an exact topological classification of such fields. More precisely, for the present, we give an exact topological classification of the flows related to these fields.

**Definition 1.** Let $P^2$ and $P'^2$ be two smooth 2-surfaces with Hamiltonian vector fields $w$ and $w'$ on them. Let $\sigma$ and $\sigma'$ be the corresponding smooth flows on $P$ and $P'$. These two flows are called exactly topologically equivalent (or exactly $C^k$-equivalent), if there exists an orientation-preserving homeomorphism (respectively, $C^k$-diffeomorphism) from $P$ into $P'$ transforming the flow $\sigma$ into the flow $\sigma'$, i.e., $\sigma' \circ \xi = \xi \circ \sigma$. In the case $k \leq 0$, $\xi$ transforms the vector field $w$ into the field $w'$.

In this paper, we shall only consider the exact topological classification of Hamiltonian flows. The case of $C^k$-equivalence will be investigated later. Let us note that, in the case of $C^k$-equivalence, we shall not assume that the equivalence mapping $\xi$ preserves the symplectic structure (i.e., $\xi$ need not be a symplectomorphism). The point is that every Hamiltonian flow can be regarded as Hamiltonian with respect to many different symplectic structures. The main object that we are concerned with is the flow (vector field) itself. From this point of view, the symplectic and Hamiltonian structures are secondary objects needed only for describing a flow (field). Thus, our goal is to classify flows (fields).

Morse function $F$ divides the surface $P$ into its level lines. Consider the so-called molecule $Y$ corresponding to this function, (see, e.g., Fig. 1). The molecule $Y$ is constructed by the same algorithm as the classical molecule $W$ in the theory of topological classification of integrable Hamiltonian systems with two degrees of freedom (for its definition and detailed description see, e.g., [3]). The only difference is that, instead of 2-tori, here we consider one-dimensional circles, and, instead of critical fibers of the Liouville foliation, we take one-dimensional critical level lines of $F$ on $P$.

The classical molecule $W$ classifies integrable systems with two degrees of freedom up to rough topological equivalence, whereas the new molecule $Y$ classifies Morse functions on 2-surfaces up to smooth equivalence.
(in other words, up to a diffeomorphism transforming level lines of the first function to those of the second one).

Let us indicate some formal distinctions between \( Y \) and the classical molecule \( W \) introduced in the theory of topological classification of integrable Hamiltonian systems with two degrees of freedom [1]. Unlike \( W \), the new molecule \( Y \) is not marked. In other words, in the theory of classification of Hamiltonian systems with one degree of freedom, the classical \( r \)-marks and \( n \)-marks do not appear. They are needed only for classification of systems with two degrees of freedom. Besides, in our case, the molecule \( Y \) contains no atoms with stars. Recall that in [1] we denoted by stars the vertices of atoms related to critical hyperbolic trajectories with nonoriented separatrix diagram (for more details, see [1, 2]). Here, obviously, we do not have their analogs.

On each edge of the molecule \( Y \), we put an arrow showing the direction of increase of the function \( F \). Thus, all the edges of the molecule acquire an orientation. Below we shall assume that all these orientations are fixed (in this situation, the molecule \( Y \) will be called oriented). Let us note that it is sufficient to put an arrow only on one edge of the molecule. Then the orientations of the other edges can uniquely be reconstructed. Under an exact topological equivalence, the oriented edges of \( Y \) must transform to oriented edges of \( Y' \). In particular, if two systems are exactly equivalent and we know the direction in which the Hamiltonian of the first system increases, then we can define the direction of increase of the other Hamiltonian.

The surface \( P \) can be regarded as a result of gluing together a number of atoms \( P^2_c \) (Fig. 1), i.e., two-dimensional surfaces with boundary satisfying the following condition. There exists a Morse function \( f \) on \( P^2_c \) and a connected graph \( K \subset P \) such that \( f \) has only one critical value \( a \) and \( K_c = f^{-1}(a) \). It is easy to see that each atom is a tubular neighborhood of the graph \( K_c \). More precisely, an atom is a pair \((P^2_c, K_c)\).

It is clear that exactly topologically equivalent Hamiltonian systems on 2-surfaces have the same oriented molecules \( Y \). Therefore, \( Y \) is a partial invariant of exact topological classification. However, that is not enough. It turns out that \( Y \) should be endowed with some more delicate “marks.” As a result, we shall obtain some object which will be a complete invariant (allowing us to distinguish equivalent and nonequivalent flows). Now we describe these additional “marks.”

\section{2. \( \Lambda \)-invariant}

Consider the restriction of the system \( w \) to a fixed saddle atom \( P^2_c = V \). Consider all critical points of the function \( F \) on the atom. By our assumptions, they are nondegenerate. For each of them, we define a number \( \Lambda(w, F) \) in the following way. Let \( \omega \) be the symplectic structure. We define the number \( \Lambda(w, F) \) related to a given critical point by setting

\[
\Lambda(w, F) = \left( -\det \left( \sum_j \omega^{ij} \frac{\partial^2 F}{\partial x^j \partial x^k} \right) \right)^{1/2},
\]

where \((x^i)\) are arbitrary regular local coordinates on the surface in a neighborhood of the critical point. This number is invariant in the sense that it does not depend on the choice of local coordinates. The point is that the expression in parentheses determines a linear operator. Therefore, its determinant does not change under coordinate transformations. In fact, \( \Lambda \) coincides with the inverse of the eigenvalue of the linear operator which is simply the linearization of the Hamiltonian field \( w \) at the critical point. Recall that, due to symplecticity, the eigenvalues \( \lambda_1 \) and \( \lambda_2 \) satisfy the condition \( \lambda_1 = -\lambda_2 \). Let us note that the number \( \Lambda \) is uniquely determined by the form \( \omega \) and the Hamiltonian \( F \) and also by the flow \( \sigma \) itself.

Let \( S_1, \ldots, S_2 \) be the critical points of \( F \), that is, the vertices of the atom \( P^2_c = V \). Thus, some real number \( \Lambda_i \) appears at every vertex \( S_i \) of the atom \( P^2_c \).

\textbf{Definition 2.} The collection of real numbers \( \{\Lambda_i\} \) (considered up to a common positive factor, i.e., up to proportionality) on the vertices of the saddle atom \( V \) is said to be the \( \Lambda \)-invariant of the Hamiltonian system \( w \) (on \( V \)). In the case of a minimax atom \( V \) (i.e., an atom of type \( A \)), the \( \Lambda \)-invariant does not appear. Determining \( \Lambda \)-invariants for each atom of the molecule, we obtain the \( \Lambda \)-invariant of the whole molecule.