We calculate the homology groups of hypersurfaces in \( \mathbb{CP}^{n+1} \), \( n \geq 3 \), with fixed number and, maybe, position of singular points and sufficiently high degree. In the case of quadratic singularities, we use the results of the calculations to give a topological description (as specific as possible) of such a hypersurface by means of decomposing it into a connected sum. In this case the topological type of the hypersurface is determined by its dimension, degree, and the number of singular points. Bibliography: 7 titles.

\[ \text{§0. Nonsingular hypersurfaces} \]

In what follows we use only the homology and cohomology with integer coefficients. We denote by \( b_n(X) \) the Betti number \( \text{rk} H_n(X) \).

Let \( X \subseteq \mathbb{CP}^{n+1} \) be a nonsingular hypersurface of degree \( d \). Its differential type is determined by \( n \) and \( d \): \( X \cong X_n(d) \). As a model hypersurface we can take that given by the Fermat equation

\[ X_n(d) = \{ x_0^d + x_1^d + \ldots + x_{n+1}^d = 0 \}. \]

The structure of \( X_n(d) \) was studied in sufficient detail by W. Browder, R. S. Kulkarni, A. S. Libgober, J. W. Wood, and others (see [3, 4] and the references cited therein). Here we mention the basic facts only. It easily follows from the Lefschetz hyperplane section theorem that if \( i \neq n \), then \( H_i X = H_i \mathbb{CP}^n \). Recall that

\[ H_i \mathbb{CP}^n = \begin{cases} \mathbb{Z}, & \text{if } i \text{ is even, } i \leq 2n, \\ 0, & \text{otherwise.} \end{cases} \]

Let \( y \) be the generator of \( H_2 X = \mathbb{Z} \). For \( n \) even let \( h \in H_n X \) be the homology class dual to \( y^{n/2} \). The class \( h \) is primitive (indivisible). Let \( n \neq 2 \). \( X_n(d) \) admits a connected sum decomposition of the form

\[ X_n(d; s) \cong M_n(d) \# a(S^n \times S^n), \]

where \( b_n(M_n(d)) = 0 \) or \( 2 \), for \( n \) odd, and \( b_n(M_n(d)) - |\text{sign } X_n(d)| \leq 5 \), for \( n \) even. It can be shown that the manifold \( M_n(d) \) is determined uniquely up to diffeomorphism (cf. [5, 6]).

§1. Diffeomorphism and rigid isotopy. Generic hypersurfaces

Below we restrict our consideration mainly by hypersurfaces with quadratic singularities, though all that follows can be transferred with appropriate changes to the case of arbitrary isolated singularities.

The following definition seems to be most suitable for the topological study of the hypersurfaces with isolated singularities.

**Definition.** Two hypersurfaces with quadratic singularities are diffeomorphic if there is a homeomorphism \( f \) between them such that

1. \( f \) is a diffeomorphism outside the singular points and
2. in the vicinities of the corresponding singular points, there are suitable holomorphic coordinates \( x_1, x_2, \ldots, x_{n+1}; y_1, y_2, \ldots, y_{n+1} \) such that the hypersurfaces are given by \( \sum_{i=1}^{n+1} x_i^2 = 0 \) and \( \sum_{i=1}^{n+1} y_i^2 = 0 \), and \( f \) is given by \( x_i = y_i, i = 1, \ldots, n + 1 \).

In other words, $f$ must be (locally) extendable to a diffeomorphism of some neighborhoods of the hypersurfaces, holomorphic in the vicinities of singular points. We easily see that a diffeomorphism is also a PL-homeomorphism with respect to suitable triangulations of the hypersurfaces.

A sufficient condition for diffeomorphism is yielded by rigid isotopy.

**Definition.** By a rigid isotopy we mean a 1-parametric family of hypersurfaces of fixed type. (It is sufficient that the Milnor numbers of singular points do not change during the isotopy.)

It would be interesting to find two hypersurfaces which are diffeomorphic but not rigidly isotopic.

In its turn, the simplest way to prove the rigid isotopy is to prove that the corresponding moduli space is irreducible. Thus, we see that all nonsingular hypersurfaces of the same dimension and degree are diffeomorphic. More generally, there are several cases where the rigid isotopy type (and so the topological and the diffeomorphism type) of a hypersurface is uniquely determined by its dimension, degree, and the number of singular points. For the simplicity we restrict our consideration by quadratic singularities only.

**Notation.** Let $P_1, \ldots, P_s \in \mathbb{CP}^{n+1}$. We define $\phi(n; \{P_1, \ldots, P_s\})$ as the minimal possible degree of a hypersurface $Y$ with quadratic singularities such that $\text{Sing}Y = \{P_1, \ldots, P_s\}$. If the points $P_1, \ldots, P_s$ are in “general position,” then the number $\phi(n; \{P_1, \ldots, P_s\})$ depends only on $n$ and $s$, and we denote it by $\phi(n; s)$.

**Proposition.** The rigid isotopy type of a hypersurface of degree $d$ with quadratic singular points $P_1, \ldots, P_s \in \mathbb{CP}^{n+1}$ is uniquely determined in each of the following cases:

1. the singular points $P_i$ are fixed,
2. the singular points $P_i$ are in generic position and the degree $d$ is sufficiently high: $d \geq \phi(n, s)$,
3. the singular points $P_i$ are arbitrary and $d > 2s$.

Case (1) was noted by Dimca [1]. Cases (2) and (3) can be proved by means of extending his arguments. In the above-listed cases (1)-(3) we will speak about generic hypersurfaces.

§2. *Estimates for the Number of Singular Points*

In dimensions more than two, the best estimates for the number of singular points are due to Varchenko [7].

**Definition.** The Arnold number $A(m, d)$ is defined by

$$A(m, d) = \text{card} \left\{ (a_1, \ldots, a_m) \in \mathbb{Z}^m : \frac{(m-2)d}{2} < \sum_{i=1}^{m} a_i \leq \frac{md}{2} \right\}.$$ 

**Proposition (Arnold’s Conjecture).** Let $X \subset \mathbb{C}^{n+1}$ be a hypersurface of degree $d$ and dimension $n$ with $s$ isolated singularities. Then $s > A(n + 1, d)$.

This estimate was shown to be exact for cubic hypersurfaces by Kalker [2] (note that $A(n+1, 3) = \binom{n+1}{n/2}$).

By certain combinatorial calculations one can deduce the following proposition.

**Proposition 2.1.** If $n$ is even, then for $(n - 4)(d - 2) > 17$, $(n, d) \neq (6, 12)$, we have the inequality $s < \min b^*_n(X_n(d)), b^*_n(X_n(d))$.

§3. *Homologically Standard Hypersurfaces*

Let $X \subset \mathbb{CP}^{n+1}$ be a hypersurface with quadratic singularities. It easily follows from the Lefschetz Hyperplane Section Theorem that if $i \neq n, n+1$, then $H_iX = H_i\mathbb{CP}^n$. Let $y$ be the generator of $H_2X = \mathbb{Z}$. For $n$ even let $h \in H_nX$ be the homology class dual to $y^{n/2}$.

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