A TYPE OF AUTOMORPHIC FORM

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The aim of this paper is to construct a \( \Gamma \)-automorphic form of weight \(-2k\) for a given finitely generated Fuchsian group \( \Gamma \) of the second kind. The construction heavily relies on important properties of the outer function introduced by Widom, which were established by the author. The constructed \( \Gamma \)-automorphic form possesses the highest possible smoothness for the weight \(-2k\). Bibliography: 8 titles.

Let \( \Gamma \) be a Fuchsian group of transformations of the unit disk \( \mathbb{D} \) onto itself. A function \( f \) analytic in \( \mathbb{D} \) is called an analytic automorphic form of weight \(-2k\), \( k \) being a positive integer, if for every \( T \in \Gamma \) the following relation holds:

\[
    f(T(z)) = f(z)(T'(z))^k.
\]

Nontrivial (i.e., not identically equal to zero) automorphic forms of weight \(-2k\) can exist only if \( \Gamma \) is a Fuchsian group of the second kind [1]. Ch. Pommerenke [2] was the first to construct an example of an analytic automorphic form of weight \(-2\) for a Fuchsian group of the second kind without parabolic elements and such that its limit set satisfies the Carleson condition. The condition of the absence of parabolic elements was eliminated in the paper of Dnase [3], where certain analytic \( \Gamma \)-automorphic forms of weight \(-2k\) were constructed for every positive integer \( k \). Since \( \Gamma \)-automorphic forms are closely related to the structure of the limit set of the group, it is interesting to give some more examples of such forms.

In the present paper finitely generated Fuchsian groups of the second kind are studied. The limit set for such groups does satisfy the Carleson condition [2]. Essentially, the new construction presented here is possible due to the regular distribution of the values \( T(0) \), \( T \in \Gamma \), as well as the lengths of complementary intervals of the limit set for the class of groups in question.

In a sense, the form we obtain turns out to have the maximal possible smoothness in the closed unit disk \( \overline{\mathbb{D}} \).

In what follows \( \Lambda^n \) denotes the class of functions \( f \) analytic in \( \mathbb{D} \) for which the condition \( |f^{(n)}(z)| \leq c_f, \quad z \in \mathbb{D}, \) is valid. The symbol \( \Lambda^{n+\alpha}, \ 0 < \alpha < 1, \) means the subclass of \( \Lambda^n \) for which the following estimate holds:

\[
    |f^{(n)}(z) - f^{(n)}(\xi)| \leq c_f|z - \xi|^\alpha, \quad z, \xi \in \overline{\mathbb{D}}.
\]

Ch. Pommerenke constructed an analytic \( \Gamma \)-automorphic form \( h \in \Lambda^1 \) of weight \(-2\). It turns out that this is the maximal possible smoothness of nonzero forms of weight \(-2\). Moreover, in general, it is possible to construct automorphic forms of weight \(-2k\) from \( \Lambda^k \), and this order of smoothness is sharp.

We recall that a closed set \( E \subset \partial \mathbb{D} \) is called a Carleson set if \( \text{mes} \, E = 0 \) and

\[
    \int \log \text{dist}(z, E) \, |dz| < \infty.
\]

**Theorem 1.** Let \( \Gamma \) be a Fuchsian group of the second kind whose action in \( \mathbb{D} \) is disconnected, let \( E \) be its limit set and let \( f \) be an analytic \( \Gamma \)-automorphic form of weight \(-2k\). Then the assumption \( f \in \Lambda^{k+\alpha}, \ 0 < \alpha < 1, \) implies \( f \equiv 0 \).

**Proof.** First of all, differentiating the relation

\[
    f(T(z)) = f(z)(T'(z))^k
\]

\( k - 1 \) times, thereafter putting \( z = 0 \), and choosing a sequence \( T_n \in \Gamma \) so that \( T_n(0) \to \xi \in E \), we find that if \( f \in \Lambda^k \), then

\[
    f|_E = f'|_E = \cdots = f^{(k-1)}|_E = 0.
\]

Furthermore, let $J$ be an arbitrary free side of the fundamental Ford polygon $D$ for the group $\Gamma$, and let $I \subset \partial \mathbb{D} \setminus E$ be a complementary arc for $E$ such that $J \subset I$.

Consider the set of images $\{T(I)\}$, $T \in \Gamma$. For every $\sigma > 0$ there exists an arc $L \subset \partial \mathbb{D}$ that contains infinitely many images $T(I)$ and whose length is less than $\sigma$. Let $I_0 = T_0(I)$, $T_0 \in \Gamma$, $I_0 \subset L$, and let $\xi_1, \xi_2$ be the endpoints of the arc $I_0$. If $f \in \Lambda^{k+\sigma}(\partial \mathbb{D})$, then using (1) and an easy modification of Lemma 1 from [4], we obtain the following relation:

$$|f(\xi)| \leq c_f \sigma^\alpha \min(|\xi - \xi_1|^k, |\xi - \xi_2|^k), \quad \xi \in I_0. \quad (2)$$

Now we consider the following function on $\partial \mathbb{D}$:

$$\rho(\xi) = \frac{|\xi - \xi_1|}{|\xi - \xi_2|},$$

which is defined by this formula if $\xi$ belongs to the interval $S \subset \partial \mathbb{D} \setminus E$ with the endpoints $\xi_1$ and $\xi_2$.

If the transformation $T(z) = e^{\alpha \frac{z-z_0}{a-z_0}}$ maps $\xi_1$ and $\xi_2$ to $t_1$ and $t_2$, respectively, then for $\xi \in S$ we have

$$\rho(T(\xi)) = \frac{|T(\xi) - t_1|}{|T(\xi) - t_2|} = \frac{|T(\xi) - T(\xi_1)|}{|T(\xi) - T(\xi_2)|} \cdot \frac{|\xi - \xi_1|}{|\xi - \xi_2|} = \frac{|\xi - \xi_1|}{|\xi - \xi_2|} \cdot \frac{1 - |a|^2}{|1 - \bar{a}\xi|^2} = \rho(\xi)|T'(\xi)|,$$

or

$$\frac{\rho(T(\xi))}{|T'(\xi)|} = \rho(\xi), \quad \xi \in \partial \mathbb{D}. \quad (3)$$

In terms of the above function $\rho$ relation (2) can be rewritten as follows:

$$|f(\xi)| \leq c_f, \sigma^\alpha \rho^k(\xi), \quad \xi \in I_0. \quad (4)$$

Recalling that $I_0 = T_0(I)$ we transform (4):

$$|f(T_0(\xi))| \leq c_f, \sigma^\alpha \rho^k(T_0(\xi)), \quad \xi \in I,$n

and furthermore

$$|f(T_0(\xi))| |T_0'(\xi)| \leq c_f, \sigma^\alpha \rho^k(T_0(\xi)) |T_0'(\xi)|^{-k}, \quad \xi \in I,$n

which together with (3) and (0) yields

$$|f(\xi)| \leq c_f, \sigma^\alpha \rho^k(\xi), \quad \xi \in I,$n

whence, since $\sigma > 0$ is arbitrary, we obtain $f|I = 0$, and so $f|J = 0$. Since $J$ is an arbitrary free side of $D$, we have $f|_{\partial \mathbb{D} \setminus E} = 0$, and then $f \equiv 0$, as required.

To construct a $\Gamma$-automorphic form of order $-2k$ we need some auxilliary functions. Put

$$u(z) = \sum_{T \in \Gamma} |T'(z)|, \quad z \in \partial \mathbb{D} \setminus E.$$

It is clear that

$$u(T_0(\xi)) = \sum_{T \in \Gamma} |T'(T_0(\xi))| = \sum_{T \in \Gamma} \frac{|T(T_0(\xi))'|}{|T_0'(\xi)||T_0'(\xi)|} = u(\xi) \frac{u(\xi)}{|T_0'(\xi)|}. \quad (5)$$

In the sequel we assume $\Gamma$ to be finitely generated, then the limit set $E$ of $\Gamma$ satisfies the Carleson condition and the function $\log u$ is summable on $\partial \mathbb{D}$ [2].