ON ONE METHOD OF FINDING THE EQUILIBRIUM STATE

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An algorithm for solving a linear “production-exchange” model is described. The model reduces to a parametric model with strictly concave utility functions. Some properties of the parametric model are studied. Since the excessive demand map satisfies the discovered preference condition, it is possible to apply the second form of Chebyshev centers. For a sufficiently small parameter, the algorithm converges to the equilibrium state of the initial linear model. Bibliography: 11 titles.

Consider an economic system consisting of \( m \) consumers and \( L \) producers. The behavior of the \( i \)-th consumer is described by the linear utility function

\[
(a_i, x_i) = \sum_{j=1}^{n} \alpha_{ij} \xi_{ij}, \quad \alpha_{ij} \geq 0, \quad x_i \in E^n_+,
\]

the reserve of products \( b_i \geq 0 \), and the share \( \delta_{il} \geq 0 \) of the \( l \)-th producer’s profit coming to the \( i \)-th consumer’s budget. Every producer is characterized by a convex closed bounded set \( \Omega_l \) of vectors \( y_l \in E^n_+ \) within the capacity of production. There are positive elements in every row and every column of the matrix \( (a_{ij}) \). It is natural to assume that \( \sum_{i=1}^{m} \delta_{il} = 1 \); for each \( i \) there exists an index \( l_i \) such that \( \delta_{il_i} > 0 \); every set \( \Omega_l \) contains a positive vector.

The linear Arrow–Debreu model \([11]\) consists of finding vectors \( x_i^* \geq 0 \), \( y_l^* \in \Omega_l \) and a vector of product prices \( p^* > 0 \) such that

\[
x_i^* = \arg \max \left\{(a_i, x_i): x_i \geq 0, \ (p^*, x_i) \leq (p^*, b_i) + \sum_{l=1}^{L} \delta_{il}(p^*, y_l), \right\}, \quad i = 1, \ldots, m; \tag{1}
\]

\[
y_l^* = \arg \max \left\{(p^*, y_l): y_l \in \Omega_l \right\}, \quad l = 1, \ldots, L; \tag{2}
\]

\[
\sum_{i=1}^{m} x_i^* = \sum_{i=1}^{m} b_i + \sum_{l=1}^{L} y_l^*. \tag{3}
\]

The vectors \( x_i^*, i = 1, \ldots, m, y_l^*, l = 1, \ldots, L, \) and \( p^* \) are called a solution of the Arrow–Debreu model or its equilibrium state.

A net-exchange linear model, which is a particular case of the linear Arrow–Debreu model, is known to be equivalent to some system of a finite number of linear equations and nonlinear (convex) inequalities \([8]\). An analogous fact was also established for the model \((1)-(3)\) \([2]\). However, in contrast to the net-exchange model, the system equivalent to the model \( (1)-(3) \) contains an infinite rather than a finite number of convex inequalities. In order to find solutions of this system (which form a convex set), one can make use of nonsmooth optimization methods \([1, 7, 10]\). In particular, in \([2]\) a cutting-off method was employed.

One more approach to finding the equilibrium state of the linear Arrow–Debreu model is associated with the fulfillment of the discovered preference condition (though the gross replaceability condition is not fulfilled for this model \([6]\)).

Let
\[ X_i(p) = \text{Arg max} \left\{ (a_i, x_i); (p, x_i) \leq (p, b_i) + \sum_{l=1}^{L} \delta_U(p, y_l(p)), x_i \geq 0 \right\}, \]
\[ y_l(p) \in Y_l(p) = \text{Arg max} \left\{ (p, y_l); y_l \in \Omega_l \right\}, \]
\[ Z(p) = \sum_{i=1}^{m} X_i(p) - \sum_{i=1}^{m} b_i - \sum_{l=1}^{L} Y_l(p). \]

**Theorem 1** [9]. If there exists a solution of the model (1)-(3) such that
\[ b_i + \sum_{l=1}^{L} \delta_i y_l^* > 0, \quad i = 1, \ldots, m, \]
then \((p^*, z) > 0, p > 0, \) for all \((z \in Z(p), p \text{ is noncollinear with } p^*)\) and \(p^*\) is the unique vector of equilibrium prices in this model.

The following algorithm [3], which is an analog of the generalized gradient-descent algorithm for minimization of a convex function [10], is based on Theorem 1.

In the initial model (1)-(3), we replace linear utility functions by strictly concave utility functions and investigate properties of the model obtained for \(\epsilon > 0\):
\[ x_i^\epsilon = \text{arg max} \left\{ (a_i, x_i) + \epsilon \sum_{j=1}^{n} \sqrt{\xi_{ij}}: x_i = (\xi_{i1}, \ldots, \xi_{in}) \geq 0, (\bar{p}^\epsilon, x_i) \leq (\bar{p}^\epsilon, b_i) + \sum_{l=1}^{L} \delta_U(\bar{p}^\epsilon, y_l^\epsilon) \right\}, \]
\[ \bar{y}_l^\epsilon = \text{arg max} \left\{ (\bar{p}^\epsilon, y_l): y_l \in \Omega_l \right\}, \]
\[ Z^\epsilon(p) = \sum_{i=1}^{m} x_i^\epsilon - \sum_{i=1}^{m} b_i + \sum_{l=1}^{L} \bar{y}_l^\epsilon. \]

Then for any \(p > 0\) we put
\[ x_i^\epsilon(p) = \text{arg max} \left\{ (a_i, x_i) + \epsilon \sum_{j=1}^{n} \sqrt{\xi_{ij}}: x_i \geq 0, (p, x_i) \leq (p, b_i) + \sum_{l=1}^{L} \delta_U(p, y_l(p)) \right\}, \]
\[ y_l(p) \in Y_l(p) = \text{Arg max} \left\{ (p, y_l); y_l \in \Omega_l \right\}, \]
\[ x^\epsilon(p) = \sum_{i=1}^{m} x_i^\epsilon(p), \quad Y(p) = \sum_{l=1}^{L} Y_l(p), \]
\[ Z^\epsilon(p) = x^\epsilon(p) - Y(p) - \sum_{i=1}^{m} b_i. \]

Put additionally
\[ \bar{b}_i^\epsilon = b_i + \sum_{l=1}^{L} \delta_i \bar{y}_l^\epsilon. \]