AN OPTIMIZATION METHOD FOR CONSTRUCTING LYAPUNOV–KRASOVSKII FUNCTIONALS IN STATIONARY LAG SYSTEMS

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A method is proposed for numerical construction of Lyapunov–Krasovskii functionals for analyzing the stability of linear stationary lag systems. The functional is constructed in the form of the sum of two quadratic sums. Positive-definite matrices of quadratic forms are found as a solution of a nonsmooth optimization problem on a convex set. Bibliography: 8 titles.

1. INTRODUCTION

One of the universal methods for analyzing systems with deviating argument is the method of Lyapunov–Krasovskii functionals [1]–[3]. Its essence consists of the construction of a positive-definite (with respect to a given metric) functional that satisfies the required conditions on the solutions of the system. The lack of a technique for constructing such functionals impedes its use in practice. Even in the case of linear stationary systems

\[ \dot{x} = Ax(t) + Bx(t - \tau), \]

where \( A \) and \( B \) are constant matrices, \( x \in \mathbb{R}^n \), \( t > 0 \), and \( \tau > 0 \) is a constant lag, the construction of such functionals reduces to the solution of a system of partial differential equations subject to special boundary conditions. Thus, a “complicated” problem reduces to an “even more complicated” one.

In this paper, we use the quadratic functional with constant matrices [3]–[4]

\[ V[x(t)] = x^T(t)Hx(t) + \int_{-\tau}^{0} x^T(t + s)Gx(t + s) \, ds \]

(1)

to obtain stability conditions. With its help only sufficient conditions for stability can be obtained. Since we deal with a lag system, the total derivative of the functional (1) has the form

\[ \dot{V}[x(t)] = -z^T(t, \tau)C[G, H]z(t, \tau), \quad z(t, \tau) = \begin{pmatrix} x(t) \\ x(t - \tau) \end{pmatrix}, \]

\[ C[G, H] = \begin{vmatrix} -A^T + H - A - G & -HB \\ -(HB)^T & G \end{vmatrix}. \]

(2)

If there exist positive-definite matrices \( G \) and \( H \) such that the matrix \( C[G, H] \) is also positive definite, then the lag system is asymptotically stable.

The functional chosen has a material drawback lying in the lack of constructive methods for finding matrices \( G \) and \( H \) and, also, in the complexity of testing the positive definiteness of the \( 2n \times 2n \) matrix \( C[G, H] \) in the case of its large dimension, \( n \). In this paper, we propose a criterion for testing the positive definiteness of the matrix \( C[G, H] \) in terms of testing some \( n \times n \) matrix \( Q[G, H] \). Using this criterion, a new optimization method is devised for finding the functional (1), which provides a means for reliably estimating the stability in the given class of functionals. Usually, the matrices \( G \) and \( H \) entering into the Lyapunov–Krasovskii functional are found in the process of solving matrix equations [3]. In this paper, \( G \) and \( H \) are solutions of an auxiliary optimization problem.

2. Auxiliary results

Denote by $L_H = \{H\}$ and $L_G = \{G\}$ the sets of symmetric positive semi-definite matrices and by $L_{G,H}$ the set of pairs of matrices $(G, H)$ for which $C[G,H]$ are positive semi-definite matrices.

**Lemma 1.** Let $A$ be an asymptotically stable matrix. If $C[G,H]$ is a positive-definite matrix, then the matrices $G$ and $H$ are also positive definite.

**Proof.** As follows from the Hurwitz criterion, the positive definiteness of $C[G,H]$ implies that $G$ and $-A^T H - HA - G$ are positive definite. Since $A$ is an asymptotically stable matrix, the positive definiteness of $G$ and $-A^T H - HA - G$ implies the positive definiteness of $H$. 

**Lemma 2.** If the set $L_{G,H}$ is not empty, then it is a convex cone.

**Proof.** Let $(G_1,H_1) \in L_{G,H}$, and let $(G_2,H_2) \in L_{G,H}$, that is, let $C[G_1,H_1]$ and $C[G_2,H_2]$ be positive semi-definite matrices. Then for any $0 < \xi < 1$

$$C[(1 - \xi)G_1 + \xi G_2, (1 - \xi)H_1 + \xi H_2] = \xi C[G_1,H_1] + (1 - \xi)C[G_2,H_2],$$

and the matrix is positive semi-definite as a linear combination with positive coefficients of positive semi-definite matrices.

Further, for any $0 < \mu < \infty$

$$C[\mu G, \mu H] = \mu C[G,H],$$

that is, $L_{G,H}$ is a convex cone. 

The analysis of stability of a lag system reduces to finding positive-definite matrices $G \in L_G$ and $H \in L_H$ such that the matrix $C[G,H]$ is positive definite. Let us replace the requirement of positive definiteness of the matrix $C[G,H]$ by the requirement of positive definiteness of the matrix

$$Q[G,H] = -A^T H - HA - G - HBG^{-1}B^T H.$$  \(3\)

**Lemma 3.** The matrix $C[G,H]$ is positive definite if and only if the matrix $Q[G,H]$ is positive definite.

**Proof.** Introduce the vector $z^T(t, \tau) = (x^T(t), x^T(t - \tau))$. As follows from the definition, the condition for positive definiteness of the matrix $C[G,H]$ is equivalent to fulfillment of the inequality

$$\min_{x(t - \tau)} \{z^T(t, \tau)C[G,H]z(t, \tau)\} > 0 \tag{4}$$

for an arbitrary value of $x(t) \in \mathbb{R}^n$. A necessary condition for minimum of the function

$$z^T(t, \tau)C[G,H]z(t, \tau) = x^T(t)(-A^T H - HA - G)x(t) - x^T(t)H B x(t - \tau) - x^T(t - \tau)B^T H x(t) + x^T(t)G x(t - \tau)$$

with respect to the variable $x(t - \tau)$ is the equality to zero of the partial derivative

$$\frac{\partial}{\partial x(t - \tau)} z^T(t, \tau)C[G,H]z(t, \tau) = 0.$$ 

Calculating this derivative, we get

$$-2B^T H x(t) + 2G x(t - \tau) = 0.$$ 

Since the matrix $G$ is non-singular, the point

$$x(t - \tau) = G^{-1}B^T H x(t)$$