BEST UPPER AND LOWER BOUNDS OF THE GENERALIZED BINOMIAL DISTRIBUTION

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The persistent interest in efficient reliability bounds for complex-structure systems keeps producing new bounds, which are characterized by low computational costs and ensure acceptable error levels for applications [1-6]. In this paper, we construct best bounds for the generalized binomial distribution (GBD) (known as "m out of n" in reliability theory), which are derived using edge-disjoint paths and cuts [6]. We also propose fundamentally new bounds based on the geometric-mean transformation of the GBD to a homogeneous BD. These new bounds have certain advantages both in terms of computational complexity and in terms of the error level (as is shown by simulation).

Litvak-Ushakov Bounds for "m out of n" System

The Litvak-Ushakov bounds [6] have the form

\[
\max_{1 \leq r \leq N} \left\{ 1 - \prod_{i \in E_n} \left( 1 - \prod_{j \in E_{a_i}} \right) \right\} \leq \min_{1 \leq s \leq M} \left\{ 1 - \prod_{i \in E_m} \left( 1 - \prod_{j \in E_{b_i}} \right) \right\},
\]

where \( N \) and \( M \) are the sets of all possible groups of edge-disjoint paths and cuts, respectively. In general, however, the search for the required paths and cuts is a fairly complex combinatorial problem. The following assertions show that for "m out of n" systems (GBD) this procedure can be substantially simplified. Note that the minimal (simple) cut [1] for this system is a set of \( n - m + 1 \) elements. The maximum number of edge-disjoint cuts in the group is the whole part of the ratio \( n/(n-m+1) \). The minimal path for the "m out of n" system is a collection of \( m \) elements, and the maximum number of edge-disjoint paths in the group is the whole part of the ratio \( n/m \). In what follows, we use the notation

\[
p(b_j) = \sum_{i \in E_{b_j}} (1 - p_{a_i}), \quad 1 \leq s \leq M,
\]

\[
p(\pi_r) = \sum_{i \in E_{\pi_r}} 1, \quad 1 \leq r \leq A.
\]

Definition 1. The collection of edge-disjoint cuts is called ranked in ascending (descending) order if

\[
p(b_{h_1}) \leq \cdots \leq p(b_{h_2}) \quad \text{or} \quad p(b_{h_2}) \leq \cdots \leq p(b_{h_1}), \quad h_1 \leq h_2 \quad \text{or} \quad h_1 \geq h_2.
\]

Definition 2. The collection of edge-disjoint cuts is called edge-ranked in ascending (descending) order if

\[
p(b_{h_1}) \leq \cdots \leq p(b_{h_2}) \quad \text{or} \quad p(b_{h_2}) \leq \cdots \leq p(b_{h_1}), \quad h_1 \leq h_2 \quad \text{or} \quad h_1 \geq h_2,
\]

\[
b_0 = \sum_{1 \leq i \leq s \leq n} i \notin E_j \quad \forall \ b_j \in E_{a_s} \quad 1 \leq s \leq M.
\]

We similarly define ranked and edge-ranked edge-disjoint paths. Note that the set of edge-ranked edge-disjoint cuts (paths) is also simply a ranked set (the converse is not true, however).

In what follows, for simplicity, \( p(\sigma_r) \) and \( p(\pi_r) \) denote the upper and lower bounds on the combinations \( \sigma_r \) and \( \pi_r \), respectively.

Proposition 1. The maximum value of the lower bounds (1) for the GBD is achieved on the set of edge-disjoint paths which are edge-ranked in descending order.
Fig. 1. Relative errors of lower bounds of GBD for $m = 3$, $n = 10$.

Proof. Necessity. Without loss of generality, we consider one of the combinations $\pi_r$, ranking its edge-disjoint paths in descending order. We assume that in general this combination is not edge-ranked. Then there exists a pair of elements such that

$$p_1 < p_2 \quad \forall i_1 \in a_1, \quad \forall i_2 \in (a_2 \cup a_3) \cup \{j_1, j_2\},$$

$$a_0 = \{1 \leq i \leq n \mid i \in a' \land a_j \in \pi_r\}. \quad (2)$$

We will show that interchanging these elements we increase the lower bound (1) on the new combination:

$$\Delta = p(\pi_r') - p(\pi_r) = (1 - p_{i_1} x)(1 - p_{i_2} y) - (1 - p_{j_1} x)(1 - p_{j_2} y) = (p_{i_2} - p_{j_2})(x - y),$$

where

$$x = \prod_{i \in a_1} p_i, \quad y = \prod_{i \in a_2} p_i, \quad z = \prod_{j \in a_3} p_j, \quad a_0 = a \cup \{a_1 \cup a_2\}.$$ 

$\pi_r'$ is the combination obtained from $\pi_r$ by interchanging the elements $p_{i_1}$ and $p_{i_2}$. From the conditions $p(a_{i_1} < p_{j_2})$ or $p_{i_1} x > p_{j_2} y$, $i_1 < i_2$, and $p_{i_1} < p_{j_2}$ we obtain the inequality $x > y$. Thus, $\Delta > 0$. Ranking the new combination in descending order and interchanging a pair of elements each time as long as condition (2) is satisfied, we obtain a set of edge-disjoint paths that are edged-ranked in descending order.

Fig. 2. Relative errors of lower bounds of GBD for $m = 6$, $n = 10$. 

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