MINIMAL INVARIANT SETS OF DYNAMIC SYSTEMS
WITH BOUNDED DISTURBANCES

V. M. Kuntsevich and B. N. Pshenichnyi

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1. INTRODUCTION

The problem of determining the minimal invariant sets of dynamic systems (linear and nonlinear, continuous and
 discrete), including systems with bounded disturbances, is more than half a century old and has been discussed in numerous
 publications. Many applied tasks in control, radio engineering, and mechanics are reducible to the solution of this problem.

The existence of a bounded invariant set (of limiting boundedness or dissipativity) is a very important property of
 dynamic systems, and its determination is a topical problem. The solution of this problem is closely linked with certain
 directions in dynamic system analysis, such as analysis of stable limiting cycles and determination of their parameters.

The first publications that focused on the problem of determination of an invariant (although not minimal invariant)
 set for linear systems with bounded disturbances were apparently [1, 2]. Analysis of invariant sets of a special kind — stable
 limiting cycles — was the subject of the seminal work of a number of leading researchers, including van der Pol, Andronov,
 Krylov, Bogolyubov, and many others.

In [2, 3] this class of problems has been analyzed using the apparatus of Lyapunov functions. Although this apparatus
 has produced significant results for nonlinear systems, it limits the potential scope of the approach, whose success entirely
 depends on successful (or unsuccessful) choice of the Lyapunov function proper. The use of known results of convex analysis
 [4] has made it possible to approach the problem from a different direction, and to obtain constructive methods of determination
 of invariant sets for some classes of dynamic systems.

2. NOTATION AND AUXILIARY RESULTS

Let \( \mathbb{R}^n \) be the \( n \)-dimensional space of column vectors \( x \) with the coordinates \( x_i, i = 1, \ldots, n \). By \( \mathbb{R}^n* \) we denote the
 conjugate space of row vectors \( \psi \) with the coordinates \( \psi_i, i = 1, \ldots, n \). Then
\[
\psi x = \sum_{i=1}^{n} \psi_i x_i, \quad \|x\|^2 = x^* x = \sum_{i=1}^{n} (x_i)^2.
\]

If \( X \) and \( Y \) are two sets in \( \mathbb{R}^n \), then \( X + Y = \{x + y: x \in X, y \in Y\} \).
Let \( M \subseteq \mathbb{R}^n \) be a convex set, and take \( W_M(\psi) = \sup_{x \in M} \{\psi x: x \in M\} \). This function is called the support function of the
 set \( M \). It is easily verified that
1) if \( M \) consists of a single point \( z \), then \( W_M(\psi) = \psi x \);
2) if \( S_r \) is a ball of radius \( r \), i.e., \( S_r = \{x: \|x\| \leq r\} \), then \( W_{S_r}(\psi) = r \|\psi\| \);
3) if \( M_1 \) and \( M_2 \) are convex sets, then \( W_{M_1 + M_2}(\psi) = W_{M_1}(\psi) + W_{M_2}(\psi) \);
4) if \( A \) is an \( n \times n \) matrix and \( M \) a convex set, then \( W_{\mathbb{R}^n A}(\psi) = W_M(\psi A) \).

The separation theorem for convex sets [4] suggests the following properties:
1) if \( M_1 \) and \( M_2 \) are closed convex sets, then \( M_1 \subseteq M_2 \) if and only if \( W_{M_1}(\psi) \leq W_{M_2}(\psi) \) for all \( \psi \);
2) if \( M \) is a closed bounded convex set, i.e., \( M \subseteq R_M S_1 \), then \( W_M(\psi) \leq R_M \|\psi\| \) (in what follows, \( R_M \) is the radius
 of the minimal ball centered at zero that includes \( M \)).
if the eigenvalues $\lambda_i$, $i = 1, \ldots, n$, of the matrix $A$ are less than 1 in absolute value, then for some constant $C$ and any number $q$, $\max_i |\lambda_i| < q < 1$, we have the inequality $\|A^n\| \leq Cq^m$, $m = 0, 1, \ldots$, where the norm $\|A\|$ of the matrix $A$ is compatible with the Euclidean vector norm. Clearly, $C \geq 1$, because $A^0 = I$, where $I$ is the identity matrix (here we use a property of $n \times n$ matrices which is well known from linear algebra [5]).

3. STATEMENT OF THE PROBLEM. MAIN THEOREM

Given is a dynamic system with an additive bounded disturbances that functions in discrete time $t = 0, 1, \ldots$.

$$x_{t+1} = Ax_t + f_t, \quad f_t \in F,$$

where $x_t \in \mathbb{R}^n$, $A$ is an $n \times n$ matrix, $F$ is a convex closed bounded set in $\mathbb{R}^n$. In what follows, we assume that the eigenvalues of the matrix $A$ are strictly less than the number $q < 1$ in absolute value.

The set $M$ is called $(A, F)$-invariant if for any initial state $x_0 \in M$ and any realization of the disturbances $f_t$, the path of system (1) remains in $M$, i.e., $x_t \in M$, $t = 0, 1, \ldots$.

Remark. If the matrix $A$ is fixed, then the phrase "the set $M$ is $(A, F)$-invariant" is replaced with the phrase "the set $M$ is $F$-invariant." If $F$ is also fixed at the given stage, then we say that the set $M$ is invariant.

THEOREM 1. Under the above assumptions, the minimal invariant set of system (1) exists and is defined by

$$M(A, F) = F + AF + A^2F + \ldots$$

Any other invariant set $M$ includes this set. If $x_0 \notin M(A, F)$, then for any $\varepsilon > 0$ the path of system (1) is eventually included in the $\varepsilon$-neighborhood of the set $M(A, F)$.

Proof. We will show that the set $M(A, F)$ is well-defined by formula (2). Any point $x \in M(A, F)$ has the form

$$x = f_0 + A f_1 + A^2 f_2 + \ldots, \quad f_t \in F,$$

and

$$\|A^n f_t\| \leq \|A^n\| \|f_t\| \leq CR_F q^t.$$ (4)

Therefore, by standard results of calculus, the series in the right-hand side of (3) is convergent. The bound (4) shows that $\|x\| \leq CR_F (1 - q)$, whence $M(A, F)$ is a bounded set. Moreover,

$$R_{M(A, F)} \leq \frac{CR_F}{1 - q}.$$ (5)

Thus, formula (2) indeed defines a set. Its convexity follows from the convexity of each term of the sum, and closure is easily checked. Now, if $x$ is defined by (3), then for $f \in F$ we have $Ax + f = f + Af_0 + A^2 f_1 + \ldots \in M(A, F)$. Hence the set $M(A, F)$ is invariant.

Now let $M$ be any other convex compact invariant set. The invariance condition implies that

$$AM + F \subseteq M.$$ (6)

In the language of support functions this relationship can be rewritten as

$$W_M(\psi A) + W_F(\psi) \leq W_M(\psi) \quad \forall \psi.$$ (7)