UPPER BOUND ON THE NUMBER OF COMPLETE MAPS†

I. N. Kovalenko

The study of the asymptotic number of complete maps for the case of permutations was begun in [1]. According to [2], a complete map of the set $A$ with the operation $\circ$ is a bijection $f: A \to A$ such that the map defined by the formula $x \mapsto x \circ f(x)$ is also a bijection. If $A$ is the set of permutations of order $n$ and $\circ$ is the operation of addition (mod $n$), then no complete maps exist for even $n$; at the same time, for odd $n$, the probability that a random permutation is a complete map will be less than $\exp\{-cn\}$ for sufficiently large $n$. In [2] it is proved that $c \geq 0.08854$.

The objective of this article is to improve the bound on $c$, specifically to show that $c \geq \ln 2/2 = 0.35$. We also give an enumeration algorithm for all complete maps in a given class with $2^{m-1/2}$ elements. This may increase by the same factor the "efficiency" of random search for a complete map.

To simplify the notation, we write $n + 1$ instead of $n$, so that $n = 2m$, where $m$ is an integer. We assume that permutations take 0 to 0, and a permutation is therefore individualized by the vector $\alpha = (\alpha(1), \ldots, \alpha(n))$, where $\alpha(i)$ is the image of the number $i$. We also consider the vector $\gamma = \gamma_\alpha = (\gamma(1), \ldots, \gamma(n))$, where $\gamma(i) = i + \alpha(i)\pmod{n + 1})$. The permutation $\alpha$ is called complete if there are no repetitions among the numbers $0, \gamma(1), \ldots, \gamma(n)$; we use $N_0$ to denote the number of complete permutations. We thus have to prove the following inequality for an arbitrary $\epsilon > 0$: $N_0 \leq n!2^{-(1+\epsilon)/2}$, $n \geq n(\epsilon)$.

Note that the total number of complete permutations (i.e., such that all $i + \alpha(i)\pmod{n + 1}$ are different and 0 may go into any number) equals $(n + 1)N_0$.

1. r-Cycles. The map $(1, \ldots, n) \mapsto \gamma$ can be represented as in Fig. 1. Thus, in our example, $\gamma(1) = 3, \gamma(2) = \gamma(3) = 1$. It is helpful to combine the numbers $2j - 1$ and $2j$ into a single cell, so that we have a total of $m$ cells.

An r-cycle is a cycle without self-intersections which is incident on $2r$ nodes in the top row and $2r$ nodes in the bottom row, so that we have a sequence of the form $ududduddu$ (u is a top node, d a bottom node). This means that the cycle goes through two bottom nodes, then through two top nodes, and so on. We moreover have the following properties in r-cycles:

1) each chain $ud$ corresponds to some $i \mapsto \gamma(i)$;
2) each chain $du$ corresponds to $\gamma(j) \mapsto j$;
3) each chain $uu$ corresponds to $2i \mapsto 2i - 1$;
4) each chain $dd$ corresponds to $j \mapsto j + 1$.

Figure 2 is an example of an r-cycle for $n = 6$.

The following proposition is of key importance.

1* If the permutation $\alpha$ has an r-cycle, then the permutation $\alpha'$ defined by replacing $2i \mapsto 2i - 1, 2i - 1 \mapsto 2i$ for any chain $uu$ of the given r-cycle defines $\gamma_\alpha' = (\gamma_\alpha(1), \ldots, \gamma_\alpha(n))$, whose components are the result of permuting the components of the vector $\gamma_\alpha$.

Indeed, in each bottom-row chain, the elements interchange their places, and no other replacement takes place.

COROLLARY. The transformation of an r-cycle takes a complete permutation $\alpha$ into a complete permutation $\alpha'$, and respectively an incomplete permutation into an incomplete permutation.

Consider the class $\overline{\alpha}$ of permutations defined in the following way. For a given permutation $\alpha^0$, the class $\overline{\alpha}$ includes, in addition to $\alpha^0$, those and only those permutations that differ from $\alpha^0$ by some transpositions $\alpha(2i) \mapsto \alpha(2i - 1), \alpha(2i - 1) \mapsto \alpha(2i)$ (not necessarily on an r-cycle). Each $\overline{\alpha}$ has $2^m$ elements.

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2. Combinatorial Bounds. If we could prove that \( \overline{\alpha} \) contains at most one complete permutation, then we would have the inequality \( N_0 \leq n!2^{-m} \). In fact, this is not so, but we can show that the number \( L(\overline{\alpha}) \) of complete permutations in the class \( \overline{\alpha} \) is bounded in some averaged sense. We start by proving some lemmas.

**Lemma 1.** Let \( \xi_r \) be the maximum number of nonintersecting \( r \)-cycles of a random permutation \( \alpha \) (i.e., a permutation chosen from a uniform distribution), and

\[
q(r, k) = P(\xi_r > k).
\]

Then we have the inequality

\[
q(r, k) \leq \frac{1}{k! r^k} \frac{m^{\lceil \frac{rk}{r} \rceil}}{n^{\lceil \frac{rk}{r} \rceil}},
\]

where \( d^{[b]} \) stands for \( a(a - 1) \cdots (a - b + 1) \).

The lemma is proved by enumerating the possible choices in the following cases:

1) choosing \( r_k \) top-row cells: \( \binom{m}{r_k} \);
2) choosing \( r_k \) nonintersecting bottom-row pairs \((j, j + 1)\): \( \binom{n - rk}{rk} \);
3) partitioning cells into \( k \) groups: \( (rk)!/(k!r!) \);
4) partitioning bottom-row pairs into \( k \) groups (with allowance for order): \( (rk)!/r!k! \);
5) choosing \( k \) \( r \)-cycles keeping \( 14 \) fixed: \( ((r - 1)!(r!))^k \);
6) successful random choice of \( 2rk \) values \( \alpha(i) \) given \( i \) and \( \gamma(i) \): with probability not exceeding \( 1/n^{[2rk]} \).

Multiplying the bounds 1-6, we obtain (1).

**Lemma 2.** For arbitrary positive \( r \) and \( k \), we have the inequality

\[
Pr(\xi_1 + \ldots \xi_r > rk) < \left( \frac{e}{2k} \right)^k.
\]

Indeed, by Lemma 1,

\[
Pr(\xi_1 + \ldots + \xi_r > rk) \leq \sum_{s=1}^{r} q(s, k) \leq \frac{1}{k!} \sum_{s=1}^{r} \frac{1}{s^{k} 2^{sk}} < \frac{1}{k! (2^k - 1)} < \left( \frac{e}{2k} \right)^k / (\sqrt{2\pi k}/2) < \left( \frac{e}{2k} \right)^k.
\]

Q.E.D.

From (2) we easily obtain the inequality

\[
Pr(\xi_1 + \ldots + \xi_r > rk) < 2^{-m}
\]

(3)