ASYMPTOTIC PROPERTIES OF A NONPARAMETRIC
INTENSITY ESTIMATOR OF A NONHOMOGENEOUS
POISSON PROCESS

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The interest in parameter estimation of Poisson processes [1, 2] is attributable to the numerous applications of these
processes in engineering, as well as in physics, biology, and geology. One of the typical cases from the point of view of
the mathematical formulation of the problem is described in [1]. When a signal \( \{ \lambda(t): 0 \leq t \leq T \} \) is transmitted by modulation of intensity \( \lambda \) of a single-mode laser, the photoelectron multiplier at the receiving end picks up the emission electrons. Let \( \hat{N}(t) \) be the number of electrons counted up to time \( t \). Then the probabilistic properties of the process \( \{ \hat{N}(t): 0 \leq t \leq T \} \) are close to those of the Poisson process \( \{ N(t): 0 \leq t \leq T \} \) with intensity function \( \lambda(t) + \lambda_0, t \in [0, T] \), where \( \lambda_0 \) is the intensity of homogeneous Poisson noise. In some cases the function \( \lambda \) is periodic with a known period \( \tau > 0 \). Additional information may be available about the function \( \lambda \), i.e., its membership in a certain class of functions. Below we consider the estimation of the periodic function \( \lambda \) from a certain class of functions using observations of the process \( \{ N(t): 0 \leq t \leq T \} \). The distribution of the Poisson process is known, and we naturally consider estimators that are best in a certain sense, specifically the maximum likelihood estimators (MLE). We give the conditions on the class of functions that ensure closeness of the estimator to the unknown function for large \( T \), i.e., produce reliable transmission of the signal \( \lambda \) in the above-described problem. We also investigate other properties of the estimator.

The application of these estimators in observations of the stationary process at the points of a nonhomogeneous Poisson
process is briefly discussed in the concluding section. The paper utilizes the procedure for the analysis of the infinite-
dimensional parameter estimator in nonlinear regression (see [7, 3, 4]). All the spaces in our study are real.

1. STATEMENT OF THE PROBLEM AND THE ESTIMATOR

Let \( X \) be the space of right-continuous piecewise-constant functions \( x: [0, + \infty) \rightarrow \mathbb{R} \), such that \( x(0) = 0 \) and the magnitude of the discontinuity at each discontinuity point equals 1. The space of functions with the same properties but with the domain of definition \([0, T]\) is denoted by \( X_T \); \( B_T \) is the \( \sigma \)-algebra generated by cylindrical sets in \( X_T \). Also let \((\Omega, \mathcal{F}, P)\) be a complete probability space. All the stochastic processes considered below are defined on this space.

**Definition 1.** Let \( \lambda: [0, + \infty) \rightarrow [0, + \infty) \) be a Lebesgue-measurable function integrable on every finite interval. The process \( \{ N(t): t \geq 0 \} \) is called a Poisson process with intensity \( \lambda \) if it has independent increments and for each \( t_1, t_2, 0 \leq t_1 < t_2 \), the increment \( N(t_2) - N(t_1) \) is Poisson-distributed with the parameter \( \int_{t_1}^{t_2} \lambda(t) dt \).

We know that the paths of the Poisson process \( \{ N(t): t \geq 0 \} \) are contained in \( X \). Note that the function \( a(t) := \int_0^t \lambda(u) du, t \geq 0, \) is a compensator of the point process from Definition 1.

We use the following assumptions.

(i) For \( T > 0 \), \( \{ N(t): t \in [0, T] \} \) is a Poisson process with a fixed but unknown intensity \( \lambda_0 \).

(ii) The function \( \lambda_0 \) is periodic with a known period \( \tau, \tau < T \).

The limit $\lambda_\omega(\tau - 0)$ exists, and the function $\lambda_\omega$ defined by the equalities $\lambda_\omega(t) = \lambda_\omega(0)$, $0 \leq t < \tau$, and $\lambda_\omega(\tau) = \lambda_\omega(\tau - 0)$, is contained in $K$ - a given subset of nonnegative functions from $C([0, \tau])$ compact in the space $(C([0, \tau]), 1)$.

Assume that $\lambda_\omega(t) = \min \{\lambda(t) \mid \lambda \in K\}$ for $t \in [0, \tau]$ and $M = \max \{\lambda(t) \mid t \in [0, \tau], \lambda \in K\}$.

(iv) For the function $\lambda_\omega$ we have $\int_0^\tau \frac{1}{\lambda_\omega(t)} dt < + \infty$ (here and in what follows Lebesgue integrals are used).

In this paper, we study the properties of the MLE for the function $\lambda_\omega$ constructed from observations of the Poisson process $\{N(t) : t \in [0, T]\}$. Note that reliable results are obtained for large $T$.

The likelihood ratio for Poisson processes is known.

**Lemma 1** [5]. Let $\{N_i(t) : t \in [0, T]\}$ be a Poisson process with intensity $\lambda_i$ and $P_i$ the measure induced by this process in the space of paths $(X_T, B_T)$, $i = 1, 2$. If $\int_0^T \frac{\lambda_1(t)}{\lambda_2(t)} dt < + \infty$, then $P_1 \ll P_2$ and for almost all (with respect to the measure $P_2$) functions $x_T \in X_T$,

$$\frac{dP_1}{dP_2}(x_T) = \exp \left( \int_0^T \frac{\lambda_1(t)}{\lambda_2(t)} dx_T(t) - \int_0^T (\lambda_1(t) - \lambda_2(t)) dt \right).$$

The first integral on the right-hand side of the equation is a Lebesgue–Stieltjes integral, and we take $0/0 : = 0$. Note that with $P_2$-probability 1 this integral is a finite sum of a random number of summands.

For $\lambda \in K$ let $P_\lambda$ be the measure on $(X_T, B_T)$ induced by the Poisson process whose intensity is identical with the function $\lambda$ on the interval $[0, \tau)$. Also let $P_0 = P_{\lambda_0}$. If conditions (i)-(iv) are satisfied, then by Lemma 1

$$\frac{dP_1}{dP_0}(x_T) = \exp \left( \int_0^T \frac{\lambda_1(t)}{\lambda_2(t)} dx_T(t) - \int_0^T (\lambda_1(t) - \lambda_2(t)) dt \right) (\mod P_0).$$

Here $\lambda$ is the function obtained by periodic continuation of $\lambda$ with period $\tau$ from the half-open interval $[0, \tau)$ to $[0, + \infty)$.

From condition (iv) we have

$$E \int_0^T |\ln \lambda_\omega(t)| dN(t) = \int_0^T |\ln \lambda_\omega(t)| d\lambda_\omega(t) dt < + \infty.$$

Therefore for each $\lambda \in K$ with probability 1

$$\int_0^T |\ln \lambda(t)| dN(t) < \int_0^T (|\ln \lambda_\omega(t)| + |\ln M|) dN(t) < + \infty.$$

To construct the MLE, it is thus sufficient to consider the functional

$$Q_T(\lambda) = -\frac{T}{T} \int_0^T \lambda(t) dt + \frac{T}{T} \int_0^T \ln \lambda(t) dN(t), \ \lambda \in K.$$

From our discussion it follows that there exists a set $\Omega_0$, $P(\Omega_0) = 1$ such that $Q_T(\lambda) \in \mathbb{R}$ for any $\lambda \in K$ and $T \geq \tau$.

For each $\omega \in \Omega_0$ the functional $Q_T(\lambda)$ is a function continuous in $\lambda$ on $K$ (this follows from the properties of the Poisson process).

**Lemma 2.** When conditions (i)-(iv) are satisfied, for each $T \geq \tau$ there exists a $K$-valued random element $\lambda_T$ that satisfies for each $\omega \in \Omega_0$ the relationship $Q_T(\lambda_T) = \max_{\lambda \in K} Q_T(\lambda)$.

The proof of the lemma is similar to the proof of existence of a measurable estimator [3]. It involves application of the theorem of measurable choice [6].

**Definition 2.** The element $\lambda_T$ from Lemma 2 is called the intensity MLE of the Poisson process $\{N(t) : t \in [0, T]\}$.

In general, the element $\lambda_T$ is not uniquely defined. In such cases, we take one of these elements as the MLE.