FRACTAL SETS DEFINED BY FINITE TRANSUDCERS

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The notion of fractal set was introduced in 1919 by F. Hausdorff, whose ideas were further developed in [1-3]. The theory of fractal sets has numerous applications [1]. Many sets on the real line or in the n-dimensional Euclidean space are fractal sets. Such are the Cantor set and related sets [3] (fractal sets associated with the Q-S-representation are also described in [3]).

In this article we consider finite R[0, 1]-transducers and investigate the conditions under which the sets generated by these transducers are fractal. An R[0, 1]-transducer is a particular form of R-transducer [4]. An arbitrary R-transducer A defines the real function \( f_A(x) \) by sequentially processing any input infinite binary representation \( x \) of the real number \( x \) into an output binary representation \( y \) so that \( f_A(x) = y \). The output superword \( y \) may contain the symbols 2 and \( \overline{2} \) (minus 2), which denote overflow. An arbitrary R-transducer A = (K, H, q0) has the state set \( K \), the instruction set \( H \), and the initial state \( q_0 \). The instruction \( pa \rightarrow bq \) defines the following action: the R-transducer reading the word \( a \) from the input tape when in state \( p \) adds the word \( b \) to the output tape and goes to state \( q \). The operation starts in the initial state \( q_0 \) when the output tape is empty and the input tape contains the input infinite binary representation of \( x \). Sequential processing of \( x \) is performed using the instructions of the R-transducer A. For an R[0, 1]-transducer A the first executed instruction has the form \( q_0 \, \overline{0} \rightarrow 0 \, \overline{0} \, q_1 \), where the symbol \( \overline{V} \) stands for the decimal point. In general, any real function can be defined by an R-transducer [4]. An R[\( m \)-transducer, unlike an R-transducer, has \( m \) input tapes and \( n \) output tapes, where \( m, n \geq 1 \). Finite R[1, 2]-transducers constructed in [5] generate analogues of the Koch snowflake and the Sierpinski napkin. These are well-known objects of fractal geometry. A finite R-transducer can be used to define a continuous nowhere differentiable function [5, 6]. Let \( Dom(f) \) be the domain of definition of the function \( f \). If \( A \) is an R[0, 1]-transducer, then it defines the set \( \{f_A(x) : x \in [0, 1] \} \).

The main result of our study can be stated as follows: if \( A \) is a finite R-transducer and the set \( Dom(f_A) \) is a nowhere dense continuum, then \( f_A \) is a fractal set in the narrow sense. The latter implies that its Hausdorff–Bezikovich dimension is a fractional number. This result is a corollary of a particular case for the R[0, 1]-transducer. We have also derived some natural generalizations of this result (e.g., for the n-dimensional Euclidean space with \( n \geq 2 \)). We have shown that the set \( Dom(f_A) \) is of measure 1 if it is everywhere dense in \( [0, 1] \) and \( A \) is a finite R[0, 1]-transducer.

The Hausdorff–Bezikovich dimension, the entropy dimension, and the \( \alpha \)-dimensional Hausdorff measure are defined in [3]. In the usual way, \( V^* \) is the set of all words in the alphabet V (the set of all finite sequences of elements of the set V), including the empty word \( e \), \( V^+ = V^* \setminus \{e\} \) is the length of the word \( v \), \( V^w \) is the set of all superwords in the alphabet V (the set of all infinite sequences of elements of the set V). We use some definitions from [4, 5]. In particular, the real number defined by the binary representation \( \alpha = 0\overline{\alpha} \), where \( \beta \in \{0, 1\}^\mathbb{N} \), is denoted by \( \overline{\alpha} \) or \( \|\alpha\| \). The symbol \( \overline{V} \) stands for the decimal point.

The R-transducer \( A = (A, H, q_0) \) is called an R[0, 1]-transducer if \( K = \{q_0, q_1, \ldots\} \) is a finite or a countable set, the set \( H \) contains the instruction \( q_0 \, \overline{0} \rightarrow 0 \, \overline{0} \, q_1 \), and for any number \( i \geq 1 \) and any symbol \( a \in \{0, 1\} \) there is an instruction of the form \( qa \rightarrow bq \), where \( j \geq 1 \), \( b \) is a symbol \( 0, 1 \) or the empty word \( e \). The R[0, 1]-transducer A defines a partial real function \( f_A : [0, 1) \rightarrow [0, 1] \). The R-transducers A and B are called equivalent if \( f_A = f_B \) [4]. Dom(\( f_A \)) is the domain of definition of the partial real function \( f_A \).

The R[0, 1]-transducer A = (K, H, q0) is called normalized if it satisfies the following conditions:
- finite or countable index sets I and J are given, which include 0, and \( K = \{q_0\} \cup \{q_i : i \in I\} \cup \{q_i^* : i \in J\} \);
- the set \( H \) contains the instruction \( q_0 \, \overline{0} \rightarrow 0 \, \overline{0} \, q_0' \);
- for every state \( q \in K \setminus \{q_0\} \) and any symbol \( a \in \{0, 1\} \) there is an instruction with the left-hand part \( qa \rightarrow q_i^* \) or an instruction of the form \( qa \rightarrow bq_i^* \), where \( b \in \{0, 1\} \).

d) for every state $g \in K$, there are words $\alpha$ and $\beta$ such that the macroinstruction $q_00\alpha \rightarrow \beta g$ is executable.

**LEMMA 1.** For every $R_{(0,1)}$-transducer $A$ there exists an equivalent normalized $R_{(0,1)}$-transducer $B$.

The proof is obvious.

The $R_{(0,1)}$-transducer $A$ is called finite if its state set is finite.

We now give a number of definitions that are used in Theorem 1 and the following lemmas. Let $A = (K, H, q_00)$ be an arbitrary fixed $R_{(0,1)}$-transducer.

For any state $p \in K \setminus \{q_00\}$, we denote by $A^p$ the $R_{(0,1)}$-transducer obtained from $A$ by replacing the instruction $q_00 0 \lor 0 \lor q1_i$ with the instruction $q_{00} 0 \lor 0 \lor \lor p$. The state $p$ is of type 1, 2, 3, or 4 if the set $\text{Dom}(f_{Ap})$ is respectively empty, finite, countable, or is a continuum.

The notation $q u \rightarrow \rightarrow p$ implies that the transducer $A$ reading the word $u$ in state $q$ goes to state $p$. If, moreover, at least one of the symbols 0 or 1 is delivered at the output, we write $q u \rightarrow \rightarrow p$. The state $p$ is called self-branching in $h$ steps if there exist different words $u_1, u_2$ of length $h$ such that $p u_1 \rightarrow \rightarrow p, i = 1, 2$.

$A \Sigma$-labeling is any function of the form $\mu : \{0, 1\}^* \rightarrow \Sigma$. Such functions are also called labelings or labeled trees. Here $\Sigma = \{0, 1, e\}$, and the set $\{0, 1\}^*$ consisting of all words in the alphabet $\{0, 1\}$, including the empty word $e$, is the set of nodes of a complete binary infinite tree $D^1$. The nodes $u, v$ are joined by an edge if $v \in \{u0, ul\}$ or $u \in \{v0, vl\}$. The node $e$ is the root of the tree.

The $R_{(0,1)}$-transducer $A$ defines two labelings $\mu_1 : \{0, 1\}^* \rightarrow K$ and $\mu_2 : \{0, 1\}^* \rightarrow \{0, 1, e\}$ in the following way: $\mu_1(e) = q_1, \mu_2(e) = e, \mu_1(v) = p, if q_00 \rightarrow \rightarrow p, \mu_2(v) = b$, if reading the word $v$ in state $q_1$ the transducer $A$ executes an instruction of the form $q_{00} \rightarrow \rightarrow bq_j$ as the last instruction and $v \neq e$.

Let $n$ be any natural number. Then the set $S(n) = \{p | q_00 \rightarrow \rightarrow p for some word $u$ of length $n\}$ is called the $n$-trace.

We assume that the pair $(m, n)$ has the characteristic $\Pi(m, n)$. There obviously exist finitely many different characteristics. Let us assign a color to each characteristic, saying that the pair $(m, n)$ is painted the color $\Pi(m, n)$. This corresponds to the terminology used in Ramsey's theory [7].

**LEMMA 2.** There exists an infinite sequence of natural numbers $n_1 < n_2 < ...$ such that all pairs $(n_i, n_j)$ with $i < j$ are painted the same color $\Pi = \Pi(n_1, n_2)$.

This lemma is a variant of the Ramsey theorem.

**THEOREM 1.** Assume that the $R_{(0,1)}$-transducer $A$ has a finite state set and $\text{Dom}(f_{Ap})$ is a continuum nowhere dense in $[0, 1]$. Then $\text{Dom}(f_{Ap})$ is a fractal set in the narrow sense.

Proof. Let $A = (K, H, q_00), E = \text{Dom}(f_{Ap})$, and $H(E)$ is the $\alpha$-dimensional Hausdorff measure of the set $E$. Below we give a number $0 < \alpha_0 < 1$ such that $H_{\alpha}(E) = 0$ for $\alpha > \alpha_0$ and $H_{\alpha}(E) = \infty$ for $0 < \alpha < \alpha_0$. Hence it follows that $H_\alpha(E) = 0$ for $\alpha = 1$, and therefore the set $E$ is of Lebesgue measure 0. The topological dimension of the set $E$ is also 0, because it is nowhere dense. The Hausdorff–Bezikovich dimension of the set $E$ is $\alpha_0$. We thus find that $E$ is a fractal set in the narrow sense. The proof relies on Lemmas 3-15.

Suppose that in accordance with Lemma 2 we have chosen the sequence of natural numbers $n_1 < n_2 < ...$ such that all the pairs $(n_i, n_j)$ with $i < j$ have the same characteristic $\Pi = \Pi(n_1, n_2)$. Here $\Theta = 0$, because the set $\text{Dom}(f_{Ap})$ is nowhere dense in $[0, 1]$. The sets $S', S_1$, and $S_{11}$ are called respectively the sets of branching, self-branching, and strongly self-branching states. The set $S'$ is nonempty, because $\text{Dom}(f_{Ap})$ is a continuum. The set $S_{11}$ is also nonempty by the following lemma.

**LEMMA 3.** For every state $p \in S'$ there exists a state $q \in S_{11}$ such that $(p, q) \in Y$.