SOME EXTREMAL PROBLEMS FOR CIRCULAR POLYGONS

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The main result of this paper is the solution of the following problem posed by J. Hersch: find the maximal conformal radius on the family of all hyperbolic polygons with \( n \) sides \((n \geq 3)\). It is proved that the maximum is attained on a regular polygon. Bibliography: 5 titles.

INTRODUCTION

A hyperbolic \( n \)-gon \( D_n \), \( n \geq 3 \), is a simply connected domain whose boundary consists of \( n \) arcs of hyperbolic lines in the unit disk \( U = \{ z : |z| < 1 \} \). As usual, a hyperbolic line is an arc of a circle orthogonal to the absolute \( T = \{ z : |z| = 1 \} \).

J. Hersch conjectured (see R. Kühnau's paper [1]) that, for fixed \( n \geq 3 \), a regular \( n \)-gon has maximal conformal radius among all hyperbolic \( n \)-gons. For triangles, this conjecture was proved in [1].

The above problem falls into the class of isoperimetric problems considered in [2–3]. The main result of the present paper is Theorem 1 below. As a particular case, it contains a complete solution of Hersch's problem. To prove Theorem 1, we apply the method used in [2] for the solution of the Polya–Szegö problem on the maximal conformal radius for a family of \( n \)-gons of fixed area.

We say that a circle \( C \) intersecting the absolute \( T \) has the \( \beta \)-property, \( 0 \leq \beta \leq 1/2 \), if the angle between \( C \) and the radius hitting the point of intersection of \( C \) and \( T \) is at most \( \beta \pi \). A circular \( n \)-gon \( D_n \subset U \) is called \( \beta \)-circular if its sides lie on circles having the \( \beta \)-property. Obviously, the \( 0 \)-circular \( n \)-gons are precisely the hyperbolic \( n \)-gons. We denote by \( D_n(\beta) \) the regular circular \( n \)-gon with vertices at the points \( \zeta_k = \exp(2\pi i(k - 1)/n), k = 1, \ldots, n \), and with angles equal to \( 2\beta \pi \), \( 0 \leq \beta \leq 1 \). Let \( R(D, z_0) \) be the conformal radius of a simply connected domain \( D \) with respect to a point \( z_0 \in D \), \( R(D) = \max_{z \in D} R(D, z) \).

Theorem 1. Let \( D_n \) be a \( \beta \)-circular \( n \)-gon, \( n \geq 3 \), \( 0 \leq \beta \leq 1/2 \). Then

\[
R(D_n) \leq \frac{2^n \Gamma(1 - \frac{2}{n})\Gamma(\frac{1}{2} + \frac{1}{n})\Gamma(\frac{1}{2} + \beta + \frac{1}{n})}{\Gamma(1 + \frac{2}{n})\Gamma(\frac{1}{2} - \frac{1}{n})\Gamma(\frac{1}{2} + \beta - \frac{1}{n})}.
\]

Here \( \Gamma(z) \) is Euler's gamma function.

Equality in (1) holds only in the case \( D_n = e^{i\theta}D_n(\beta) \) with \( \theta \) real.

Let \( D_1, D_2 \) be nonoverlapping simply connected domains on the sphere \( \mathbb{C} \), and let \( 0 \in D_1 \), \( \infty \in D_2 \). The classical inequality of M. F. Lavrent'ev

\[
R(D_1, 0) \cdot R(D_2, \infty) \leq 1
\]

has been generalized and improved by several authors. For instance, in [2] a sharp estimate of the product of the conformal radii of domains \( D_1 \) and \( D_2 \) separated by an \( n \)-link polygonal line was obtained. Here we shall find an analog of (2) for domains \( D_1, D_2 \) separated by a curve consisting of \( n \) circular arcs.

Theorem 2. Let \( D_n \) be a \( \beta \)-circular \( n \)-gon with vertices on the absolute \( T \), and let \( 0 \in D_n \), \( \bar{D}_n = \mathbb{C} \setminus \bar{D}_n \), \( n \geq 3 \), \( 0 \leq \beta < 1/2 \). Then

\[
R(D_n, 0)R(\bar{D}_n, \infty) \leq 2^n \frac{\Gamma^2(1 - \frac{2}{n})\Gamma^2(\frac{1}{2} + \frac{1}{n})\Gamma(\frac{1}{2} + \beta + \frac{1}{n})\Gamma(\frac{3}{2} - \beta - \frac{1}{n})}{\Gamma^2(1 + \frac{2}{n})\Gamma^2(\frac{1}{2} + \frac{1}{n})\Gamma(\frac{1}{2} + \beta - \frac{1}{n})\Gamma(\frac{3}{2} - \beta + \frac{1}{n})}.
\]

Equality in (3) is attained only in the case \( D_n = e^{i\theta}D_n(\beta) \) with \( \theta \) real.

The proofs of Theorem 1 and 2 are given in Sec. 2. In Sec. 1, an explicit expression of the reduced module of a circular triangle with respect to its geometric vertex is obtained. This expression is used to prove the main results in Sec. 2.

§ 1. REDUCED MODULE OF A CIRCULAR TRIANGLE

By a (topological) triangle we mean a simply connected domain $D$ with three marked distinct boundary elements $e_0, e_1, e_2$. The definition of the reduced module $m(D; e_0|e_1, e_2)$ of a triangle $D$ with respect to its vertex $e_0$ was introduced in [2]. This definition is similar to the definition of the reduced module $m(D, z_0)$ of a simply connected domain $D$ with respect to its point $z_0$. The relation

$$m(D; e_0|e_1, e_2) = \frac{1}{\angle(e_1e_2e_0)} \log R(D; e_0|e_1, e_2), \quad e_0 \neq \infty,$$

connects the reduced module of a triangle with the angular conformal radius introduced by J. Hersch in [4].

Consider the circular triangle $\Delta(\alpha, \beta, \gamma)$ bounded by the line segments $[0, 1]$ and $[0, re^{i\alpha\pi}]$ (where $0 < \alpha < 1, r = r(\alpha, \beta, \gamma) > 0$) and an arc of some circle passing through the points $z_1 = 1$ and $z_2 = re^{i\alpha\pi}$. Here we assume that $\Delta(\alpha, \beta, \gamma)$ has angle $\beta\pi, 0 \leq \beta \leq 1$, at its vertex $z_1$ and $\arg(z_2 - 1) = \pi(1 - \beta - \gamma)$, where $0 \leq \arg(z_2 - 1) < \pi$.

Further, consider the function

$$z = f_0(\zeta) = c\left(-\zeta, F(\alpha + \gamma, 1 - \beta - \gamma; 1 + \alpha; \zeta)\right),$$

where $-\pi < \arg(-\zeta) < 0$; $F(\cdot, \cdot, \cdot, \cdot; \cdot)$ is the Gauss hypergeometric function defined by its expansion for $|\zeta| < 1$; and

$$c = e^{i\alpha\pi} \frac{\Gamma(1 - \gamma)\Gamma(\alpha + \beta + \gamma)\Gamma(1 - \alpha)}{\Gamma(1 - \alpha - \gamma)\Gamma(\beta + \gamma)\Gamma(1 + \alpha)}.$$  

(1.1)

The function $f_0(\zeta)$ maps the upper half-plane $H = \{\zeta : \text{Im}\zeta > 0\}$ onto the triangle $\Delta(\alpha, \beta, \gamma)$, and $f_0(0) = 0, f_0(1) = 1, f_0(\infty) = z_2$ (see, for instance, [5], p. 244 of the Russian edition). If $\zeta \to 0$, then (1.1) yields the asymptotic formula

$$f_0(\zeta) = c(-\zeta)^{\alpha}(1 + O(\zeta)).$$

(1.3)

Using (1.2), (1.3), and Lemma 1 from [2] (the latter describes how the reduced module of a triangle changes under conformal mappings), we get

$$m(\Delta(\alpha, \beta, \gamma); 0|z_1, z_2) = \frac{1}{\alpha \pi} \log \frac{4^\alpha \Gamma(1 - \gamma)\Gamma(\alpha + \beta + \gamma)\Gamma(1 - \alpha)}{\Gamma(1 - \alpha - \gamma)\Gamma(\beta + \gamma)\Gamma(1 + \alpha)}.$$  

(1.4)

Replacing $\alpha$ by $2\alpha$ in (1.4) and putting $\gamma = \frac{1}{2} - \alpha - \beta$, we obtain a formula for the reduced module of the isosceles circular triangle $\Delta(\alpha, \beta) = \Delta(2\alpha, \beta, 1/2 - \alpha - \beta)$:

$$m(\Delta(\alpha, \beta); 0|z_1, z_2) = (2\alpha^2 \pi)^{-1} F(\alpha, \beta).$$

(1.5)

Here and below

$$F(\alpha, \beta) = \alpha \log \frac{2^\alpha \Gamma(1 - 2\alpha)\Gamma(\frac{1}{2} + \alpha)\Gamma(\frac{1}{2} + \beta + \alpha)}{\Gamma(1 + 2\alpha)\Gamma(\frac{1}{2} - \alpha)\Gamma(\frac{1}{2} + \beta - \alpha)}.$$  

(1.6)

It should be noted that equality (1.5) with $\beta = 1/2 - \alpha$ yields a formula for the reduced module of an isosceles Euclidean triangle, which was used in [2] for solving the isoperimetric problem of Polya-Szegö. For $\beta = 1/2$, we have

$$m(\Delta(\alpha, 1/2); 0|z_1, z_2) = 0,$$

which means that the reduced module of each sector of the unit disk is zero.

Using (1.5) and the equality case of Theorem 3 from [2], we obtain the following expression for the reduced module of a regular circular $n$-gon:

$$m(D_n(\beta), 0) = (n/2\pi)F\left(\frac{1}{n}, \beta\right).$$

(1.7)