APPROXIMATION ON THE LIMIT CONTINUUM OF A DEGENERATE KLEINIAN GROUP

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Let $\Gamma$ be a degenerate Kleinian group with limit continuum $K$. Then the linear combinations of the fractions $\frac{\xi - T(a_j)}{T \xi}$, $T \in \Gamma$, $j = 1, \ldots, n(\Gamma)$, are dense in $C(K)$ and $\lambda^\alpha(K)$. Bibliography: 6 titles.

Let $\Gamma$ be a degenerate Kleinian group (see [1, Chap. 9, G]), i.e., an analytically finite Kleinian group whose unique invariant component $\Omega$ is simply connected. Assume that $\infty \in \Omega$; then the limit set $K$ of $\Gamma$ is a bounded continuum without interior and is invariant under $\Gamma$. We can define natural classes of functions on $K$: $C(K)$ is the set of continuous functions on $K$; $\lambda^\alpha(K)$ ($0 < \alpha < 1$) is the Hölder class consisting of all functions $f$ satisfying $|f(z) - f(\xi)| = o(|z - \xi|^\alpha)$ as $|z - \xi| \to 0$. The norm in $\lambda^\alpha$ is given by

$$
\|f\|_{\lambda^\alpha} \overset{\text{def}}{=} \|f\|_{C(K)} + \sup_{z, \xi \in K} \frac{|f(z) - f(\xi)|}{|z - \xi|^\alpha}.
$$

Moreover, as in [2] by E. M. Dyn'kin, we can introduce the spaces $\lambda^\alpha(K)$ for $0 < m < \alpha < m + 1$. Namely, $\lambda^\alpha(K)$ consists of functions $f$ for which there exist the (suitably defined) derivatives $f'(z), \ldots, f^{(m)}(z)$, $z \in K$, and $f^{(m)} \in \lambda^{\alpha - m}(K)$.

It is well known that, for a continuum $K$ with empty interior, the rational functions with poles off $K$ are dense in $C(K)$ (I have not come across a similar statement for $\lambda^\alpha(K)$ in the literature). It turns out that, for the above specific continuum $K$, one can even construct rational functions forming a dense subset of $C(K)$ and of $\lambda^\alpha(K)$ and having poles at points of a fixed countable set naturally related to $K$.

Theorem 1. Let $D_{\rho_0} = \{\xi \in C \setminus K : \text{dist}(\xi, K) \geq \rho_0\}$. (a) There exists $n = n(\rho_0, \Gamma, K)$ such that, if some points $a_1, \ldots, a_n \in D_{\rho_0}$ are not $\Gamma$-equivalent, then the linear combinations of the functions $\frac{1}{T(a_j)}$, $T \in \Gamma$, $j = 1, \ldots, n$, are dense in $C(K)$.

(b) Under the same assumptions, the linear combinations of the functions

$$
\frac{1}{T(\xi) - a_j}, \quad T \in \Gamma, \quad j = 1, \ldots, n,
$$

are dense in $C(K)$.

Theorem 2. Let $\alpha > 0$ be a noninteger. There exists $n = n(\rho_0, \Gamma, K, \alpha)$ such that, if some points $a_1, \ldots, a_n \in D_{\rho_0}$ are not $\Gamma$-equivalent, then the linear combinations of the functions $\frac{1}{\xi - T(a_j)}$, $T \in \Gamma$, $j = 1, \ldots, n$, are dense in $\lambda^\alpha(K)$.

Before proving Theorems 1 and 2, we establish another statement, which is probably of independent interest. We recall that, by Mergelyan's theorem (see [3]), the polynomials in $z$ are dense in $C(K)$. As far as I know, a similar assertion for $\lambda^\alpha(K)$ has not yet occurred in the literature.

Lemma 1. Let $K$ be a bounded continuum with empty interior and connected complement, and let $\alpha > 0$ be a noninteger. Then the polynomials in $z$ are dense in $\lambda^\alpha(K)$.

Proof. First, we consider the case $0 < \alpha < 1$. For $f \in \lambda^\alpha(K)$, let $\omega_k(f; \delta) = \sup_{z, \xi \in K} |f(z) - f(\xi)|$ ($z, \xi \in K$, $|z - \xi| < \delta$) be the modulus of continuity of $f$. We extend $f$ to the entire complex plane $C$ in such a way that $\omega(\delta, \delta) = \sup_{z, \xi \in K} |f(z) - f(\xi)|$ (see [4, Chap. 1, §7]). By definition, $\omega(\delta) = o(\delta^\alpha)$, $\delta \to 0$. We put

$$
f_\delta(z) = \frac{1}{\pi \delta^2} \int_{|\xi - z| < \delta} f(\xi) d\sigma_\xi.
$$

Assume that we are given \( \varepsilon, 0 < \varepsilon < 1 \). Fixing \( \delta \) so that
\[
\tau^{2\alpha} \overset{\text{def}}{=} \sup_{\sigma \leq \delta} \frac{\omega(\sigma)}{\sigma^\alpha} < \varepsilon^2,
\]
we obtain for \( |z_1 - z_2| \geq \tau \delta \) (since \( |f(z) - f_\delta(z)| \leq \omega(\delta) \)):
\[
\frac{|(f_\delta(z_1) - f(z_1)) - (f_\delta(z_2) - f(z_2))|}{|z_1 - z_2|^{\alpha}} \leq \frac{2 \omega(\delta)}{\tau^{2\alpha}} \leq \frac{2 \tau^{2\alpha}}{\tau^\alpha} = 2 \varepsilon.
\] (1)

But if \( |z_1 - z_2| < \tau \delta \), then
\[
\frac{|(f_\delta(z_1) - f(z_1)) - (f_\delta(z_2) - f(z_2))|}{|z_1 - z_2|^{\alpha}} \leq \frac{|f(z_1) - f(z_2)|}{|z_1 - z_2|^{\alpha}} + \frac{|f_\delta(z_1) - f_\delta(z_2)|}{|z_1 - z_2|^{\alpha}} \leq \frac{2 \omega(|z_1 - z_2|)}{|z_1 - z_2|^{\alpha}} < 2 \varepsilon^2,
\] (2)
i.e., \( f_\delta \to f \) in \( \lambda^\alpha(K) \).

Denoting by \( P(K, \alpha) \) the closure of the polynomials in \( \lambda^\alpha(K) \), we see, by the above, that it suffices to verify the relation \( f_\delta \in P(K, \alpha) \) for all \( \delta > 0 \). Let \( D \supset K \) be a domain bounded by a smooth curve \( L \), and let \( K_\delta = \{ \xi \in \mathbb{C} \setminus K : \text{dist}(\xi, K) < \eta \} \). Applying the Green formula, we obtain
\[
f_\delta = f_1 + f_2 + f_3 + f_4, \quad \text{where } f_1(z) = \frac{1}{2\pi i} \int_L \frac{f(z)}{z - \zeta} d\zeta,
\]
\[
f_2(z) = -\frac{1}{\pi} \int_D \frac{\partial f_\delta}{\partial \xi} \frac{d\sigma_\xi}{\xi - z}, \quad f_3(z) = -\frac{1}{\pi} \int_{K_\delta} \frac{\partial f_\delta}{\partial \xi} \frac{d\sigma_\xi}{\xi - z}, \quad f_4(z) = -\frac{1}{\pi} \int_{D \setminus (K_\delta \cup K)} \frac{\partial f_\delta}{\partial \xi} \frac{d\sigma_\xi}{\xi - z}.
\]

Since \( \text{mes}_2 K_\eta \to 0 \) as \( \eta \to 0 \) and \( \partial f_\delta / \partial \eta \) is bounded, it can easily be seen that
\[
\|f_\delta\|_{C(K)} \to 0 \quad \text{and} \quad \omega_C(f_\delta; \delta) \leq \omega(1) \frac{\delta}{\delta},
\]
whence, by analogy with (1) and (2), we derive that \( \|f_\delta\|_{\lambda^\alpha(K)} \to 0 \). Thus, it suffices to prove that \( f_1, f_2, f_4 \in P(K, \alpha) \). We fix \( \eta > 0 \) and choose a \( C^2 \)-smooth Jordan curve \( L \) lying in \( K_\eta \) and containing \( K \) strictly inside. We formulate a property of the polynomials that were used in [5] (this property has not been singled out earlier).

**Sublemma 1.1.** Let \( G \) be the inside of \( L \), and let \( v \) be a function analytic in \( G \) and continuous in \( \overline{G} \). Then the polynomials \( P_N \) defined in [5] have the following properties:
\[
|v(z) - P_N(z)| \leq c\omega_0 \left( \frac{1}{N} \right), \quad z \in \overline{G}, \quad \omega_0(\delta) = \omega_G(v; \delta); \quad (3)
\]
\[
\omega_G(P_N; \delta) \leq c\omega_0(\delta). \quad (4)
\]

Here (3) is directly taken from [5], and (4) is proved in a similar way.

Applying Sublemma 1.1 to \( v = f_1 \) and \( v = f_4 \) (for them \( \omega_0(\delta) \leq C \delta \)), we find, as in (1) and (2), that \( f_1, f_4 \in P(K, \alpha) \). It remains to analyze \( f_2 \).

Let \( Q \) be a square containing \( K \). We split it into congruent squares \( q_j (j = 1, \ldots, M) \) with side \( \sigma \), and put
\[
\mu_j = -\frac{1}{\pi} \int_{q_j} \frac{\partial f_\delta}{\partial \xi} d\sigma_\xi, \quad \nu_j = -\frac{1}{\pi} \int_{q_j} (\xi - o_j) \frac{\partial f_\delta}{\partial \xi} d\sigma_\xi, \quad j = 1, \ldots, M,
\]
where \( o_j \) is the center of \( q_j \). Observe that
\[
|\mu_j| \leq c\sigma^2, \quad |\nu_j| \leq c\sigma^3, \quad j = 1, \ldots, M. \quad (5)
\]