CERTAIN EXAMPLES OF BIRATIONALLY RIGID VARIETIES WITH A PENCIL OF DOUBLE QUADRICS

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We prove that double covers of $P^1 \times Q$, where $Q \subset P^{k+1}$ is a nondegenerate quadric, and the branch divisor cut out by a hypersurface of the type $(2\lambda, 2k - 2)$ are birationally superrigid. In particular, they have only one structure of a Fano fibration. Therefore, they are nonrational.

Introduction

The aim of the present paper is to prove that certain higher-dimensional varieties with a pencil of Fano double quadrics are birationally superrigid. The double quadrics themselves were studied in [3]. Informally speaking, we prove a very particular case of the following "general principle":

if a Fano fibration is "sufficiently twisted" over the base, then "the birational geometry of the entire variety reduces to the birational geometry of the generic fiber," that is, a Fano variety over a nonclosed field.

All the known examples of birationally rigid Fano fibrations [4, 5] can be considered as realizations of this principle. Here we give one more example.

The paper consists of three sections. In Sec. 1, the necessary definitions are collected, and we also prove some elementary general facts. In Sec. 2, we give an explicit construction of a certain family of algebraic varieties. After that, we formulate the principal result, that is, birational superrigidity of any variety from this family, and start to prove it. Section 3 contains the key part of the proof, that is, exclusion of infinitely near maximal singularities. We do this by means of a technique that is a higher-dimensional generalization [4] of the method of V. A. Iskovskikh and Yu. I. Manin [1].

The result was obtained during my stay at Max-Planck-Institut für Mathematik in Bonn. I would like to thank the staff of the Institute for their hospitality.

1. Fano Fibrations and Their Birational Correspondences

Here we introduce a large class of higher-dimensional algebraic varieties which are (at least, in principle) to be studied by means of the method of maximal singularities.

Definition 1. A projective algebraic variety $V$ equipped with a morphism

$$\pi: V \to S$$

is said to be a Fano fibration if the following conditions are satisfied:

(i) $V$ is nonsingular in codimension 1:

$$\text{codim} \text{Sing } V \geq 2;$$

(ii) the dimension of the base $S$ is strictly smaller than $\dim V$;

(iii) the fiber $F_\eta$ over the generic (nonclosed) point of the base $S$ is irreducible, and canonical adjunction terminates on $S$, i.e., for any Weil divisor $D$ on $F_\eta$ there is $\alpha \in \mathbb{R}_+$ such that the linear system

$$|aD + bK_{F_\eta}|$$

is empty for any nonnegative integers $a, b \in \mathbb{Z}_+$ such that $b > \alpha a$. 

Obviously, the (principal) condition (iii) is equivalent to the following condition: for a general point \( t \in S \), canonical adjunction terminates on the corresponding fiber \( F_t = \pi^{-1}(t) \).

**Definition 2.** Let \( D \subseteq V \) be a Weil divisor on the Fano fibration \((V, \pi)\). The least \( \alpha \in \mathbb{R}_+ \) for which
\[ aD + bK_V = 0 \]
for any \( a, b \in \mathbb{Z}_+ \) such that \( b > \alpha a \) is said to be the *threshold of canonical adjunction* (or, briefly, just the threshold) and denoted by \( c(D) = c(V, D) \).

**Examples of Fano fibration.** (i) Let \( S = \ast \) be a point, \( V \) be a smooth (or with mild singularities) Fano variety, i.e., the anticanonical class \((-K_V)\) is ample. For instance, \( V_M \subseteq \mathbb{P}^M \) is a smooth hypersurface of degree \( M \). Here \( \text{Pic} V = \mathbb{Z}K_V \).

(ii) **Standard conic bundles**
\[ \pi: V \rightarrow S, \]
\( V, S \) are smooth, \( E = \pi_*\mathcal{O}_V(-K_V) \) is a locally free sheaf of rank 3 on \( S \), \( V \twoheadrightarrow \mathbb{P}(E) \), and each fiber \( \pi^{-1}(t) \) is a conic in \( \mathbb{P}^2 = \mathbb{P}(E_2) \), and the following condition is satisfied:
\[ \text{Pic} V = \mathbb{Z}K_V \oplus \pi^*\text{Pic} S. \]

(iii) **Del Pezzo fibrations**
\[ \pi: V \rightarrow \mathbb{P}^1, \]
where \( V \) is smooth, \( \text{Pic} V = \mathbb{Z}K_V \oplus \pi^*\text{Pic} \mathbb{P}^1 \), and the generic fiber \( F_0 \) is a Del Pezzo surface of degree \( d \) over the nonclosed functional field \( \mathbb{C}(\mathbb{P}^1) \).

One more example of a Fano fibration will be given in Sec. 2 below. It comprises the subject of the present paper.

The general task of our theory is to describe birational correspondences between Fano fibrations.

**Definition 3.** A pair \((V', H')\), where \( V' \) is a Fano fibration, \( \pi': V' \rightarrow S' \), and \( H' \) is a Weil divisor on \( V' \), is said to be a *test pair*, if the linear system \( |H'| \) is free in codimension 1. The threshold of the divisor \( H' \) is said to be the threshold of the pair \((V', H')\), and is denoted by \( c(V', H') \).

**Examples of test pairs.** (i) Let \( V' \) be a Fano variety, \( |H'| \subseteq |-rK_V| \) be a linear system with no fixed components, and \( r \in \mathbb{Q}_+^* \). Obviously, \( c(V', H') = r \).

(ii) Let \( \pi': V' \rightarrow S' \) be a Fano fibration over a positive-dimensional base, \( |H'| \) be the inverse image of a linear system on the base, which is free in codimension 1. Obviously, \( c(V', H') = 0 \).

Assume that there is a birational map
\[ \chi: V \dashrightarrow V', \]
where \( V \) is a Fano fibration and \((V', H')\) is a test pair. The dimensions \( \dim V \) and \( \dim V' \) are assumed to be equal. Denote by \( |\chi| \) the proper inverse image of the linear system \( |H'| \) on \( V \) with respect to \( \chi \). It is a linear system of divisors on \( V \) with no fixed components. Let \( D \in |\chi| \) be a general member, so that \( |\chi| \subseteq |D| \). The threshold \( \eta(\chi) = c(V, D) \) is an important characteristic of the birational map \( \chi \).

The group of birational automorphisms of the variety \( V \) is denoted by \( \text{Bir} V \); the group of biregular automorphisms is denoted by \( \text{Aut} V \).

**Definition 4.** For a Fano fibration \( \pi: V \rightarrow S \), the group of birational automorphisms of the generic fiber
\[ \text{Bir} F_0 \subseteq \text{Bir} V \]
is said to be the group of *proper* birational automorphisms of the Fano fibration.

The thresholds of canonical adjunction accumulate, from the viewpoint of birational geometry, the most essential information about linear systems. Correspondingly, if we know how the thresholds change when we apply a birational map, then we know practically everything about this map. Now we introduce the key concept of the theory.

**Definition 5.** A Fano fibration \( V \) is said to be *birationally rigid* if for any test pair \((V', H')\) and any birational map
\[ \chi: V \dashrightarrow V' \] 987