A FAST DIRECT METHOD OF SOLVING HERMITIAN FOURTH-ORDER FINITE-ELEMENT SCHEMES FOR THE POISSON EQUATION

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The method of separation of variables (the Fourier method) is widely applied in solving difference boundary-value problems [1]. The popularity of this method is due primarily to the fact that in combination with the fast discrete Fourier transform it makes it possible to construct an algorithm with an asymptotic number of arithmetic operations close to optimal.

The author [2, 3] has used the method of separation of variables to construct algorithms for solving the grid schemes of the finite-element method with second-order precision for the Poisson equation in a rectangular domain. The method used in these papers can be applied to Lagrangian and Hermitian finite-element schemes of arbitrary order of precision. In the present paper we give generalizations of this method using the example of Hermitian fourth-order finite-element schemes.

\section{Statement of the Problem}

It is required to solve a system of linear algebraic equations

\[ \tilde{A}z = \tilde{B}f, \]  

where \( \tilde{A} = A_2 \otimes B_1 + \delta B_2 \otimes A_1, \tilde{B} \in \{ E_2 \otimes E_1, E_2 \otimes B_1, B_2 \otimes B_1 \}, \otimes \) stands for the direct (tensor) product [4, p. 223], \( E_k \) is the \( 3n_k \times 3n_k \) identity matrix, and \( A_k = M_k(\alpha) \) and \( B_k = M_k(\beta) \) are \( 3n_k \times 3n_k \) positive definite matrices, \( k = 1, 2 \), of the form

\[ M_k(\gamma) = \begin{bmatrix} T_\gamma & S_\gamma & P_\gamma \\ S_\gamma^T & R_\gamma & Q_\gamma \\ P_\gamma^T & Q_\gamma^T & G_\gamma \end{bmatrix}, \quad F_\gamma = \begin{bmatrix} P_\gamma \\ Q_\gamma \end{bmatrix}, \quad H_\gamma = \begin{bmatrix} T_\gamma & S_\gamma \\ S_\gamma^T & R_\gamma \end{bmatrix}. \]  

Here the blocks \( T_\gamma, R_\gamma, \) and \( G_\gamma \) are square matrices of \( n_k - 1, n_k + 1, \) and \( n_k \) rows respectively, and the blocks \( S_\gamma, P_\gamma, \) and \( Q_\gamma \) are rectangular matrices of dimension \((n_k - 1) \times (n_k + 1), (n_k - 1) \times n_k, \) and \((n_k + 1) \times n_k\) respectively. In addition

\[ T_\gamma = \begin{bmatrix} \gamma_1 & \gamma_3 \\ \gamma_3 & \gamma_1 \end{bmatrix}, \quad S_\gamma = \begin{bmatrix} -\gamma_5 & 0 & \gamma_5 \\ -\gamma_5 & 0 & \gamma_5 \\ -\gamma_5 & 0 & \gamma_5 \end{bmatrix}, \quad R_\gamma = \begin{bmatrix} \gamma_2 & \gamma_4 & \gamma_4 \\ \gamma_4 & \gamma_2 & \gamma_4 \\ \gamma_4 & \gamma_2 & \gamma_4 \end{bmatrix}, \quad P_\gamma = \begin{bmatrix} \gamma_6 & \gamma_6 \\ \gamma_6 & \gamma_6 \end{bmatrix}, \quad G_\gamma = \begin{bmatrix} \gamma_8 \\ \gamma_8 \\ \gamma_8 \end{bmatrix}. \]

The system of equations (1), (2) arises in approximating the Dirichlet problem for the Poisson equation in a rectangle \( \Omega = (0, l_1) \times (0, l_2) \) by the finite-element method with a fourth-degree Hermitian basis on the uniform grid

\[ \Omega_k = \{ (x_1^{(k)}, x_2^{(k)}; x_k^{(k)} = i_k h_k, i_k = \overline{1, n_k - 1}, x_i^{(k+1)} = (i_k - 1/2) h_k, i_k = \overline{1, n_k}, h_k = l_k/n_k, k = \overline{1, 2} \}. \]
In this case the quantity $\delta$ is defined by the relation

$$\delta = \frac{h_2^2}{h_1^2}.$$ 

\section{The Eigenvalue Problem}

To construct an algorithm for solving the system (1) we need to know the eigenvalues of the matrix $A_k$ relative to $B_k$. In this section we determine these eigenvalues as the roots of suitable algebraic equations of degree four.

Consider the eigenvalue problem

$$Az = \lambda Bz,$$ 

(3)

where $A = M(\alpha)$ and $B = M(\beta)$ are positive-definite $3n \times 3n$ matrices of the form (2).

We introduce the square matrices $E$ and $R_\gamma$ with $n + 1$ and $n - 1$ rows respectively and having the following forms:

$$E = \begin{bmatrix} \frac{1}{2} & 1 & \cdots & 1 \\ & \ddots & \ddots & \ddots \\ & & 1 & \frac{1}{2} \end{bmatrix}, \quad R_\gamma = \begin{bmatrix} \gamma_2 & \gamma_4 & \cdots & \gamma_4 \\ \gamma_4 & \gamma_2 & \cdots & \gamma_4 \\ \cdots & \cdots & \cdots & \cdots \\ \gamma_4 & \gamma_2 & \cdots & \gamma_4 \end{bmatrix}.$$ 

We use the notation

$$\nu_k = \sin \frac{k\pi}{n}, \quad \kappa_k = \cos \frac{k\pi}{n}.$$ 

Lemma 1. Suppose there exist inverse matrices to $R_\beta$ and $\tilde{R}_\beta$. Then the following equality holds:

$$S_\alpha \tilde{R}_\beta^{-1} E = R_\beta^{-1} S_\alpha.$$ 

(4)

Proof. We set $P = R_\beta^{-1} S_\alpha$, $\tilde{P} = S_\alpha \tilde{R}_\beta^{-1} E$, and we introduce the vectors $y^{(k)} = (y_0^{(k)}, \ldots, y_n^{(k)})^T$, $k = 0, n$, and $x^{(k)} = (x_1^{(k)}, \ldots, x_{n-1}^{(k)})^T$, $k = 1, n - 1$, with components

$$y_i^{(k)} = \kappa_i, \quad i = 0, n, \quad k = 0, n; \quad x_i^{(k)} = \nu_i, \quad i = 1, n - 1, \quad k = 1, n - 1.$$ 

It is not difficult to verify that

$$S_\alpha y^{(k)} = \lambda_k^{(1)} x^{(k)}, \quad k = 1, n - 1,$$

$$S_\alpha y^{(0)} = S_\alpha y^{(n)} = 0,$$

$$R_\beta x^{(k)} = \lambda_k^{(2)} x^{(k)}, \quad k = 1, n - 1,$$

$$\tilde{R}_\beta y^{(k)} = \lambda_k^{(3)} y^{(k)}, \quad k = 0, n,$$

where

$$\lambda_k^{(1)} = -2\alpha_k \nu_k, \quad k = 1, n - 1,$$

$$\lambda_k^{(2)} = 2\beta_k \kappa_k + \beta_2, \quad k = 1, n - 1,$$

$$\lambda_k^{(3)} = 2\beta_k \kappa_k + \beta_2, \quad k = 0, n.$$ 

We now take an arbitrary vector $y = (y_0, \ldots, y_n)^T$. For this vector we find the numbers $\xi_i, i = 0, n$, that are the coefficients in the expansion

$$y = \sum_{k=0}^n \xi_k y^{(k)}.$$ 

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