ASYMPTOTIC ANALYSIS OF THE BEHAVIOR OF A MULTICHANNEL QUEUEING SYSTEM FUNCTIONING IN A MARKOV MEDIUM

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The asymptotic behavior of some multidimensional characteristics of two Markov queueing systems, in which an incoming flow of units and their service time depend on a small parameter $\varepsilon$ and the state of the Markov medium where these queueing systems function, is investigated. Bibliography: 6 titles.

Every so often, to assess the reliability of a system, the distribution of such characteristics as time to the first failure in the system and the flow of occurrences of the system in nonoperational states needs to be known. The asymptotic behavior of the distributions of such characteristics of some complex systems was investigated in [1-4].

In this paper, the weak convergence of the processes describing the flow of occurrences of a system in nonoperational states and the flow of refused units to Poisson processes is proved for the systems under consideration.

Consider a time-continuous Markov process $\eta_{\varepsilon}(t)$, $t \geq 0$, with a state set $E = \{1, 2, \ldots, r\}$. Let the state set $E$ be divided into disjoint state subsets $E_0, E_1, E_2, \ldots, E_m$ ($E = \bigcup_{i=0}^{m} E_i$).

Denote by $X_{\varepsilon}(l)$, $l = 0, 1, 2, \ldots$, the Markov chain (MCH) embedded in the process $\eta_{\varepsilon}(t)$ and having a matrix of transition probabilities $P_{\varepsilon} = \|\lambda_{\varepsilon}^{-1}(k) \times \lambda_{\varepsilon}(k, n)\|$ and a stationary distribution $\rho_{\varepsilon} = \{\rho_{\varepsilon}(k), k = 1, \ldots, r\}$; here $\lambda_{\varepsilon}(k, n)$ is the intensity of the transition probabilities from $k$ to $n$ and $\lambda_{\varepsilon}(k) = \sum_{n \neq k} \lambda_{\varepsilon}(k, n)$.

We assume that there exists $P = \lim_{\varepsilon \to 0} P_{\varepsilon}$, $P = \|p(k, n)\|$.

Suppose that the following condition, condition A, is fulfilled.

1. $E_0$ is a state set, to which there corresponds in the limit a single essential class of states.

2. The transition probabilities $p_{\varepsilon}(i, k)$, $i \in E_n, k \in E_l$, of the embedded MCH $X_{\varepsilon}(l)$, $l = 0, 1, 2, \ldots$, are of the form

$$p_{\varepsilon}(i, k) = \begin{cases} p(i, k) - o(1), & \text{if } n = 1, \ldots, m, l < n, \\ \varepsilon p(i, k) + o(\varepsilon), & \text{if } n = 0, \ldots, m, l = n + 1, \\ 0, & \text{otherwise}. \end{cases}$$

We assume that $E_{m+1} = E_m$.

3. $\lim_{\varepsilon \to 0}(\lambda_{\varepsilon}(k)\alpha_{\varepsilon})^{-1} = \Delta_k$, $\Delta_k < \infty$, $k \in E$, where

$$\alpha_{\varepsilon} = \sum_{k \in E} q_{\varepsilon}(k)\lambda_{\varepsilon}^{-1}(k),$$

that is, $\alpha_{\varepsilon}$ is the mean sojourn time of the process $\eta_{\varepsilon}(t)$ in a state averaged over the stationary distribution of the embedded chain.

4. $\eta_{\varepsilon}(0) = j$, $j \in E_0$.

It is worth noting that, in the terminology introduced in [5] and [6], the subsets $E_i$, $i = 1, \ldots, m$, form a sequence of $\varepsilon$-nonrecurrent state subsets. Structures of this type may be used to describe a wide class of Markov queueing systems and models in reliability theory.

Let us introduce the following notation: $P_{\varepsilon}(E_s, E_t)$ is the matrix of transition probabilities of the embedded chain $X_{\varepsilon}(l)$ from states $k \in E_s$ to states $n \in E_t$, $s, i = 0, \ldots, m$; $P_{0} = \|p(k, n)\|$, $k, n \in E_0$; $P_{\varepsilon} = \|p(k, n)\|$, $k \in E_{s-1}, n \in E_s$, $s = 1, \ldots, m+1$; $\bar{q}(E_s) = \{q_{\varepsilon}(i), i \in E_s\}$, $s = 0, \ldots, m$, $\bar{q}(E_0) = \{q(k), k \in E_0\}$, where $q(k)$ is the stationary probability distribution of the MCH with the matrix of transition probabilities $\|p(k, n)\|$, $k, n \in E_0$.

Lemma. If condition A is fulfilled, then for $s = 0, \ldots, m$, we have

$$\tilde{q}_e(E_s) = \{e^s Q + o(e^s), k \in E_s\},$$

where $\{Q_k, k \in E_s\} = \tilde{q}(E_0) P_1 \cdots P_s$.

Proof. For $q_e(E_0)$ the assertion is obvious since $q_e(k) = q(k) + o(1)$, $k \in E_0$, where $q(k)$ may be found, as usual, from the system of equations for the stationary distribution of the MCH given by the stochastic matrix $P_0$. For $s = 1, \ldots, m$, we have

$$q_e(E_s) = q_e(E_{s-1}) P_e(E_{s-1}, E_s) + \sum_{k=s+1}^{m} q_e(E_k) P_e(E_k, E_s).$$

From (1) it is not hard to obtain that $q_e(k) = o(e^s)$ if $k \in E_s$, $s = 1, \ldots, m$, and hence for $s = 1, \ldots, m$, we have

$$q_e(E_s) = \tilde{q}_e(E_{s-1}) P_e(E_{s-1}, E_s) + o(e^s)$$

$$= \tilde{q}(E_0) P_e(E_0, E_1) P_e(E_1, E_2) \cdots P_e(E_{s-1}, E_s) + o(e^s) = \tilde{q}(E_0)e^s P_1 P_2 \cdots P_s + o(e^s).$$

The lemma is thus proved.

Denote by $\nu_e(k, t), k = 1, \ldots, r$, the number of occurrences of the process $\eta_e(t)$ in the state $k$ during the time $[0, T]$.

Theorem 1. If condition A is fulfilled, then

$$\{\nu_e(k, t), k \in E_m\} \underset{\text{weakly}}{\rightarrow} \{\nu_0(k, t), k \in E_m\}, \quad t \in [0, T],$$

where $\{\nu_0(k, t), k \in E_m\}$ is a homogeneous process with independent increments, and, moreover,

$$M \exp \left\{ \sum_{k \in E_m} z_k \nu_0^{(k)}(t) \right\} = \exp \left\{ ct \left( e^{-1} \sum_{k \in E_m} e^{iz_k} \sum_{l \in E_{m-1}} Q(l)p(l, k) - 1 \right) \right\},$$

$$C = \sum_{l \in E_{m-1}} Q(l) \sum_{k \in E_m} p(l, k).$$

Let us consider a queueing system consisting of $N$ sources generating units and $n$ service channels. Suppose that the whole system functions in a random medium controlled by an ergodic MCH $Z(t) \geq 0$ with the state set $U = \{1, \ldots, r\}$ and the matrix of intensities of transition probabilities $(a_{ij}, i, j = 1, \ldots, r, a_{ii} = \sum_{i \neq j} a_{ij})$. If the Markov process $Z(t)$ is in a state $i$, then the probability that an active source will generate a unit on a time interval $(t, t + h)$ is $\lambda(i, e)h + o(h)$. If at the instant a unit is generated and at least one of the service channels is unoccupied, then the unit starts to be serviced. Otherwise, the unit queues up. We shall consider the “first-in-first-out” discipline (FIFO). If at time $t$ a channel is occupied with servicing a unit and $Z(t) = i$, then the probability that the service will be completed in this channel on a time interval $(t, t + h)$ is equal to $\mu(i)h + O(h)$. All $n$ service channels are assumed to be of the same type, that is, the service intensity depends on the random medium $Z(t)$ and does not depend on the number of a channel.

At a given moment of time no more than one unit from a fixed source can be serviced. To put it differently, if a fixed source has generated a unit, then it can generate the next unit only after completion of servicing of the previous one. At the completion of servicing, a unit immediately returns to its source, whereupon the source regains its activity and becomes capable of generating the next unit for servicing.

All random variables describing the behavior of the above model, as well as the MCH $Z(t)$, are mutually independent.