MULTIVARIATE CAUCHY DISTRIBUTIONS AS LOCALLY GAUSSIAN DISTRIBUTIONS

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In the present paper, we propose a definition of locally Gaussian probability distributions of random vectors based on the linearization of their conditional quantiles. We prove that the Cauchy distribution in $\mathbb{R}^n$ is locally Gaussian and give explicit formulas for the vectors of expectations and covariance matrices of locally Gaussian approximations. We show that locally Gaussian approximations with different dimensionalities are in some sense compatible: all of them have equal corresponding correlation coefficients. For the Cauchy distribution in a Hilbert space we prove a limit theorem on the convergence of squared finite-dimensional conditional quantiles to the stable Lévy distribution.

Before we formulate the definition of a locally Gaussian distribution, we introduce the required notation. $F_{1...n}(x_1, \ldots, x_n)$ and $f_{1...n}(x_1, \ldots, x_n)$ will respectively stand for the distribution function and the density of the random vector $\xi = (\xi_1, \ldots, \xi_n)$; by $G_{1...n}(x_1, \ldots, x_n)$ and $g_{1...n}(x_1, \ldots, x_n)$ we respectively denote the Gaussian distribution function and density with the vector of expectations $m = (m_1, \ldots, m_n)$ and covariance matrix $B = [b_{ij}]$. $F^{-1}(\cdot)$ is the function inverse to a distribution function $F(\cdot)$. The superscript $^{-1}$ denotes that the element under it is omitted. The conditional density and distribution function of a random variable $\xi_k$ with respect to $\xi_1, \ldots, \xi_{k-1}, \xi_{k+1}, \ldots, \xi_n$ will be respectively denoted $f_{k|1...k-1,k+1...n}(x_k | x_1, \ldots, \hat{x}_k, \ldots, x_n)$ and $F_{k|1...k-1,k+1...n}(x_k | x_1, \ldots, \hat{x}_k, \ldots, x_n)$. The conditional quantile of order $\alpha \in [0, 1]$ will be denoted $q_{k|1...k...n}^{F}(x_1, \ldots, \hat{x}_k, \ldots, x_n)$ so that

$$F_{k|1...k...n}(q_{k|1...k...n}(x_1, \ldots, \hat{x}_k, \ldots, x_n) | x_1, \ldots, \hat{x}_k, \ldots, x_n) \equiv \alpha.$$

We will deal with random vectors whose distribution functions possess the following property.

**Property A.** The joint distribution possesses a density which is everywhere positive and differentiable together with all its marginals.

Now we give the definition of a locally Gaussian distribution.

**Definition 1.** Let $F_{1...n}(x_1, \ldots, x_n)$ be a distribution and $z^0 = (x^0_1, \ldots, x^0_n) \in \mathbb{R}^n$. If there exists a Gaussian distribution $G_{1...n}(x_1, \ldots, x_n)$ such that

1. for any $k = 1, \ldots, n$

$$F_{k|1...k...n}(x_k^0 | x_1, \ldots, \hat{x}_k, \ldots, x_n) = G_{k|1...k...n}(x_k^0 | x_1, \ldots, \hat{x}_k, \ldots, x_n),$$

2. for any $k = 1, \ldots, n$ the graphs of conditional quantiles of the Gaussian distributions $G_{k|1...k...n}(x_k | x_1, \ldots, \hat{x}_k, \ldots, x_n)$ passing trough the point $x^0 = (x^0_1, \ldots, x^0_n)$ are tangent spaces to the graphs of conditional quantiles to the graphs of the distributions $F_{k|1...k...n}(x_1, \ldots, \hat{x}_k, \ldots, x_n)$ passing through the same point,

3. $f_{1...n}(x^0_1, \ldots, x^0_n) = g_{1...n}(x^0_1, \ldots, x^0_n),$

then the distribution $F_{1...n}(x_1, \ldots, x_n)$ is called locally Gaussian at the point $x^0 = (x^0_1, \ldots, x^0_n)$. Since this is so, the distribution $G_{1...n}(x_1, \ldots, x_n)$ is called a locally Gaussian approximation to $F_{1...n}(x_1, \ldots, x_n)$ at the point $x^0 = (x^0_1, \ldots, x^0_n)$. A distribution which is locally Gaussian at every point of $\mathbb{R}^n$ is called locally Gaussian in $\mathbb{R}^n$.

Before proving that the multivariate Cauchy distribution is locally Gaussian, we will write out the explicit formulas for the conditional quantiles of the Cauchy and Gaussian distributions. First consider the Cauchy distribution in $\mathbb{R}^n$ defined by the density

$$f_{1...n}(x_1, \ldots, x_n) = \frac{\Gamma((n + 1)/2)}{(\pi)^{(n+1)/2} \prod_{k=1}^{n} \lambda_k} \left(1 + \sum_{k=1}^{n} \frac{x_k^2}{\lambda_k^2}\right)^{(n+1)/2}, \quad \lambda_i > 0, \quad i = 1, \ldots, n.$$
Simple calculations of the integrals using the B-function yield

\[ F_{x_1...x_n}(x_i \mid x_1, ..., x_i, ..., x_n) = \begin{cases} 1 - \frac{1}{2} B \left( \frac{1 + \sum_{k=1, k \neq i}^{n} x_k^2 / \lambda_k^2}{1 + \sum_{k=1}^{n} x_k^2 / \lambda_k^2}; \frac{n-1}{2} \right), & x_i \geq 0; \\ \frac{1}{2} B \left( \frac{1 + \sum_{k=1, k \neq i}^{n} x_k^2 / \lambda_k^2}{1 + \sum_{k=1}^{n} x_k^2 / \lambda_k^2}; \frac{n-1}{2} \right), & x_i < 0; \end{cases} \]  

where \( B(\cdot, n/2, 1/2) \) is the beta-distribution with parameters \( n/2 \) and \( 1/2 \) (see [1, p. 269]).

From (2) we find the equations for the conditional quantiles of the Cauchy distribution (1) passing through the point \( x^0 = (x_1^0, ..., x_n^0) \):

\[ q_{x_1...x_n}^{F}(x_1,..., x_i, ..., x_n) = F_{x_1...x_n}(x_i \mid x_1, ..., x_i, ..., x_n), \]

or

\[ q_{x_1...x_n}^{F}(x_1,..., x_i, ..., x_n) = x_i^0 \left[ 1 + \sum_{k=1, k \neq i}^{n} (x_k^0 / \lambda_k^2) \right] / \left( 1 + \sum_{k=1}^{n} (x_k^0 / \lambda_k^2)^2 \right), \quad i = 1, ..., n. \]  

Then the equations for the tangent planes to the graphs of conditional quantiles at the point \( x^0 = (x_1^0, ..., x_n^0) \) have the form

\[ x_i = x_i^0 + \sum_{j=1, j \neq i}^{n} \frac{\partial q_{x_1...x_n}^{F}(x_1,..., x_i, ..., x_n)}{\partial x_j} (x_j - x_j^0), \]

\[ = x_i^0 + \sum_{j=1, j \neq i}^{n} \frac{x_i^0 x_j^0}{\lambda_i^2} \left( 1 + \sum_{k=1, k \neq i}^{n} (x_k^0 / \lambda_k^2)^2 \right) (x_j - x_j^0). \]  

Now we come to the equations for the conditional quantiles of the Gaussian distribution \( G_{x_1...x_n}(x_1, ..., x_n) \) with the vector of expectations \( m = (m_1, ..., m_n) \) and covariance matrix \( B = \{b_{ij}\} \)

\[ G_{x_1...x_n}(x_1 \mid x_1, ..., x_i, ..., x_n) = G_{x_1...x_n}(x_i \mid x_1, ..., x_i, ..., x_n), \]

\[ i = 1, ..., n. \]  

Denoting \( G_{x_1...x_n}(x_i^0 \mid x_1^0, ..., x_i^0, ..., x_n^0) \) by \( \pi_i^0 \), we write these equations in the form (see [1, p. 346])

\[ \Phi \left( \frac{x_i - m_i + \sum_{j=1, j \neq i}^{n} \frac{b_{ij}}{\sqrt{\lambda_i}} (x_j - m_j)}{\sqrt{\lambda_i}} \right) = \pi_i^0, \]

\[ i = 1, ..., n, \]  

where \( [b_{ij}] = B^{-1} \) and \( \Phi(\cdot) \) is the standard Gaussian distribution. Therefore,

\[ x_i^0 - m_i + \sum_{j=1, j \neq i}^{n} \frac{b_{ij}}{\sqrt{\lambda_i}} (x_j - m_j) = \frac{1}{\sqrt{\lambda_i}} q(\pi_i^0), \]

\[ q(\pi_i^0) = \Phi^{-1}(\pi_i^0), \quad i = 1, ..., n. \]

In the sequel we will use the following statement.

**Lemma.** If for a Gaussian distribution in \( \mathbb{R}^n \) one knows:

(1) the values of conditional probabilities \( \pi_i^0 \) at a point \( x^0 = (x_1^0, ..., x_n^0) \):

\[ G_{x_1...x_n}(x_i^0 \mid x_1^0, ..., x_i^0, ..., x_n^0) = \pi_i^0, \quad i = 1, ..., n, \]

(2) the equations for the conditional quantiles passing through \( x^0 = (x_1^0, ..., x_n^0) \),

\[ q_{x_1...x_n}^{G}(x_1,..., x_i, ..., x_n) = x_i^0 - \sum_{j=1, j \neq i}^{n} d_{ij}(x_j - x_j^0), \]

\[ \Phi \left( \frac{x_i - m_i + \sum_{j=1, j \neq i}^{n} \frac{b_{ij}}{\sqrt{\lambda_i}} (x_j - m_j)}{\sqrt{\lambda_i}} \right) = \pi_i^0, \]

\[ q(\pi_i^0) = \Phi^{-1}(\pi_i^0), \quad i = 1, ..., n. \]