LIMIT THEOREMS FOR GLOBAL MEASURES OF THE DEVIATION OF A KERNEL ESTIMATE OF INTENSITY FUNCTION OF AN INHOMOGENEOUS POISSON PROCESS

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Limit theorems are proved for some global measures of the deviation of a kernel estimate of a periodic intensity function of an inhomogeneous Poisson process. Namely, maximal normalized absolute deviation and averaged square deviation are considered.

Introduction

Let \( N(t) \) be an inhomogeneous Poisson process on the positive semiaxis with periodic intensity function \( \lambda(t) \). Consider the problem of statistical estimation of \( \lambda(t) \) from the process realization \( N(t) \) on the segment \([0, T]\), where \( T \to \infty \). The kernel estimate for the case where \( \lambda(t) \) has a known period equal to \( \tau \) is presented in [1].

Introduce a bounded function \( K(u), u \in \mathbb{R}_1 \), with finite support \([-A, A]\), satisfying the condition \( \int K(u) \text{d}u = 1 \). Then the estimate \( \hat{\lambda}(t) \) for \( \lambda(t) \) is [1, 2]

\[
\hat{\lambda}(t) = \frac{\tau}{T \varphi_T} \left( \sum_{i=1}^{\lfloor T/\tau \rfloor} \int_{(i-1)\tau}^{i\tau} K \left( \frac{s - (i-1)\tau - t}{\varphi_T} \right) \text{d}N(s) + \int_{\tau N}^{T} K \left( \frac{s - \tau N - t}{\varphi_T} \right) \text{d}N(s) \right),
\]

where \( N = \lfloor T/\tau \rfloor \), \( \varphi_T \to 0 \), and \( T \varphi_T \to \infty \). The consistency, asymptotic normality, and asymptotic efficiency of estimate (1) are demonstrated in the same paper for any segment contained in the interval \((0, \tau)\) under the assumption that \( \lambda(t) \) is smooth enough.

The present paper studies the limit distributions of the functionals

\[
\max_{t \in [a, b]} \frac{|\lambda_T(t) - \lambda(t)|}{\lambda(t)^{1/2}},
\]

\[
\int_a^b \frac{(\lambda_T(t) - \lambda(t))^2}{\lambda(t)} \text{d}t,
\]

where \( 0 < a < b < \tau \).

Our main results are similar to those from [2, 3] for nonparametric estimators of probability density.

1. Main Results

Assume that the following conditions hold:

A1: \( K(u) \) is a symmetric function having the bounded support \([-A, A]\),

\[
K(A) \neq 0, \quad \int_{-\infty}^{\infty} K(u) \text{d}u = 1, \quad \int_{-\infty}^{\infty} K^2(u) \text{d}u = \sigma^2 < \infty,
\]

and absolutely continuous on \([-A, A]\) simultaneously with \( K'(u) \);

A2: \( \lambda(t) \) is a continuous, positive, bounded function with period \( \tau \), and \( (\lambda(t))^{1/2} \) is absolutely continuous with the bounded derivative;

A3: \( \lambda(t) \) has continuous second derivative on \([0, \tau]\).

Consider the functions \( r(t) = \sigma^{-2} \int K(t + u)K(u) \text{d}u \) and \( \omega_T \) such that \( \omega_T > 0 \) and \( \omega_T \to 0 \) for \( T \to \infty \). Let \( B \) be the intersection of all cones with vertices at \((0; 1)\) on the plane \((t, r(t))\) and bases on the vertical \( t = \omega_T \), such that \( r(t) \) is contained in the cones. Denote the angle at the vertex of \( B \) as \( \theta_T \). It is obvious that \( \theta_T \to 0 \) for \( T \to \infty \) [4, §1].


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**Theorem 1.1.** Let $0 < a < b < 1$. If conditions A1-A3 are satisfied and $\varphi_T = T^{-\delta}$, $\delta \in (1/5, 1/2)$, then there exists a constant $C : 0 < C < \infty$, such that

$$\sup_{t_T} \left| \max \left\{ \frac{T^{1/2}(\lambda_T(t) - \lambda(t))^2}{(\sigma(t))^{1/2}(\lambda(t))^{1/2}} \right| \leq t_T + (x + \log K) \theta_T^{-1} \right\} - e^{-2e^{-2}} \leq C \left( \theta_T + \frac{(\log \log T)^2}{\theta_T \log T} \right),$$

where $t_T$ is the root of the equation

$$\frac{(b - a)T^4}{(2\pi)^{1/2}} t \exp \left( -\frac{t^2}{2} \right) = 1,$$

$$K = K^2(A)/\sigma^2.$$

**Theorem 1.2.** Let the conditions A1-A3 be satisfied and let $\varphi_T = T^\delta$, $\delta \in (2/3, 1/2)$. Then

$$T^{2/3} \left\{ \int_a^{b/2} \frac{\lambda(t)}{\lambda(t)} \frac{\lambda_T(t) - \lambda(t)}{\lambda(t)} dt \frac{T^{1-\delta}}{\tau} - \sigma^2(b-a) \right\} \xrightarrow{D} N \left( 0, 2(b-a) \int \left( \int K(x+y)K(x) dx \right)^2 dy \right)$$

as $T \to \infty$, where $N$ denotes the normal distribution.

2. Approximations

Consider the stochastic process

$$Y_T^{(1)}(t) = \frac{1}{\sigma} \left( \frac{T \varphi_T}{\sigma(t)} \right)^{1/2} (\lambda_T(t) - E\lambda_T(t)), \quad t \in [0, \tau],$$

and fix the number of realizations of the Poisson process on the segment $[0, T]$. By the total probability formula we have

$$\mathbb{P}\{ \max_{t \in [s, t]} |Y_T^{(1)}(t)| < x \} = \sum_{n=0}^{\infty} \mathbb{P}\{ \max_{t \in [s, t]} |Y_T^{(1)}(t)| < x \mid N[0, T] = n \} \mathbb{P}\{ N[0, T] = n \}$$

$$= \left( \sum_{n=0}^{\infty} \mathbb{P}\{ \max_{t \in [s, t]} |Y_T^{(1)}(t)| < x \mid N[0, T] = n \} \Lambda_T e^{-\Lambda_T} / n! \right),$$

where $\varrho_T = T^{1/2+\alpha}$, $\alpha \in (0, 1/6)$, $\Lambda_T = \int_0^T \lambda(u) du$.

It follows from the Cramér theorem [5] that the second term in (6) does not exceed $C_1 \exp(-C_2 T^{2\alpha})$, where $C_1, C_2$ are absolute positive constants.

Introduce the distribution function $F(s)$:

$$F(s) = \begin{cases} 0, & s < 0, \\ \Lambda_s / \Lambda_T, & 0 \leq s \leq \tau, \\ 1, & s > \tau. \end{cases}$$

By virtue of the known property of the Poisson process, the first term in (6) is equal to

$$\sum_{|n-\Lambda_T| \leq T\tau} e^{-\Lambda_T} \frac{\Lambda_T}{n!} \mathbb{P}\left\{ \max_{i \in [s, \tau]} \left| \sum_{i=1}^n K \frac{X_i - \tau}{\varphi_T} \left( \frac{r}{\varphi_T f(t)} \right)^{1/2} \left[ \sum_{i=1}^n K \frac{X_i - \tau}{\varphi_T} - n \int_0^T K \frac{f(s)}{\varphi_T} ds \right] \right| < x \right\},$$

where $f(s)$ is the density of $F(s)$, equal to 0 outside of the segment $[0, \tau]$, and $X_i, i = 1, \ldots, n$, are independent identically distributed random variables with distribution function $F(s)$. Denote the process under the absolute value sign by $Y_{n,T}^{(1)}(t)$ and introduce the approximating process.