STRESS STATE OF A CYLINDRICAL SHELL
LOADED ALONG SEGMENTS OF THE DIRECTING CIRCLE

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Simple closed expressions for forces and moments in a circular cylindrical shell loaded uniformly by radial forces along segments of the directing circle are obtained by asymptotic synthesis methods (ASM) in [1, 2]. In practice, the pressure distribution can be significantly nonuniform, especially when a shell is in contact with other rigid bodies [3, 4]. For this reason, the results of [1, 2] are extended below to the case of a nonuniform distribution of external load. Unlike in [5], where a two-dimensional Fourier transform was employed, the solution in our case is constructed using an ASM. This method gives simple compact relations for forces and moments for a sufficiently general external load.

We denote the thickness and radius of the shell by \( h \) and \( R \), and the elastic modulus and Poisson’s ratio of the shell material by \( E \) and \( \nu \). We assume that the cross section \( z = 0 \), to which an external load is applied, is sufficiently distant from the shell edges, so that the influence of the edges on local bending can be ignored. That is, the shell is considered infinite in the axial direction. This simplifying assumption is acceptable for a self-balanced external loading which is cyclically symmetrical about the angular coordinate \( \beta \) composed of \( k \) normal forces applied periodically along the directing circle. Each of the forces \( P \) is distributed along a circular arc with length \( 2\beta_0 R \) by the law \( f(\beta) = f(-\beta) \), so that

\[
P = R \int_{-\beta_0}^{\beta_0} f(\beta) \, d\beta = 2R \int_{0}^{\beta_0} f(\beta) \, d\beta.
\]

As in [1, 2], we represent the local stress state as the sum of the principal state and a simple boundary effect. We describe the former by the Schorer type equation [6]

\[
\frac{\partial^4 \Phi}{\partial \alpha^4} + c^2 \frac{\partial^2 \Phi}{\partial \beta^2} = \frac{R^2}{Eh} p(\alpha, \beta),
\]

where \( \alpha = x/R, \ c^2 = h^2/(12(1 - \nu^2)R^2) \), \( \Phi = \Phi(\alpha, \beta) \) is a resolving function, and \( p(\alpha, \beta) \) is the density of the external radial load.

In the principal state, the bending moments \( G_1^p \) and \( G_2^p \) and the tangential forces \( T_1^p \) and \( T_2^p \) are expressed in terms of the function \( \Phi \) as

\[
\begin{align*}
G_1^p &= \nu G_2^p = -\nu \frac{D}{R^2} \frac{\partial^2 \Phi}{\partial \beta^2}, \\
D &= \frac{Eh^3}{12(1 - \nu^2)} = Ehc^2R^2, \\
T_1^p &= -\frac{Eh}{R} \frac{\partial^4 \Phi}{\partial \alpha^2 \partial \beta^2}, \\
T_2^p &= 0.
\end{align*}
\]

The stress state of the simple boundary effect is given by the equation [1]

\[
\frac{\partial^4 w}{\partial \alpha^4} + c^{-2} w = R^2 D^{-1} p(\alpha, \beta),
\]

which is written with respect to the lateral deflection (or radial displacement) \( w = w(\alpha, \beta) \). This corresponds to the force factors

\[
\begin{align*}
G_1^b &= \nu G_2^b = -\nu \frac{D}{R^2} \frac{\partial^2 w}{\partial \beta^2}, \\
T_2^b &= -\frac{Eh}{R} w, \\
T_1^b &= 0.
\end{align*}
\]
Note that here and below the superscripts \( p \) and \( b \) indicate that a factor belongs to the principal state or to the boundary effect, respectively.

The external-load density is given by  
\[
p(\alpha, \beta) = q(\beta) R^{-1} \delta(\alpha - 0).
\]
Here
\[
q(\beta) = q\left(\beta \pm \frac{2\pi}{k}\right) = \begin{cases} f(\beta) & \text{for } \beta \in [-\beta_0, \beta_0], \\ 0 & \text{for } \beta \in \left[-\frac{\pi}{k}, -\beta_0\right] \cup \left(\beta_0, \frac{\pi}{k}\right), \end{cases}
\]
and \( \delta(\alpha - 0) = \pi^{-1} \int_0^\infty \cos \lambda \alpha \, d\lambda \) is the Dirac function.

We expand the even \( 2\pi/k \)-periodic function in a cosine series:
\[
q(\beta) = \sum_{n=0}^{\infty} q_n \cos kn\beta. \tag{3}
\]
The coefficients are written as the integrals
\[
q_n = \frac{2k}{\pi} \int_0^{\beta_0} f(\beta) \cos kn\beta \, d\beta. \tag{4}
\]
For each of the harmonics \( n \), we construct solutions of Eq. (1) that decrease at \( \pm \infty \). Such a solution does not exist for \( n = 0 \). For this reason, the axisymmetric component is taken into account only in the boundary effect, while for the principal state we assume
\[
\frac{\partial^4 \Phi}{\partial \alpha^4} + c^2 \frac{\partial^2 \Phi}{\partial \beta^2} = \frac{R}{\pi E h} \sum_{n=1}^{\infty} q_n \cos kn\beta \int_0^\infty \cos \lambda \alpha \, d\lambda.
\]
This equation has the solution
\[
\Phi(\alpha, \beta) = \frac{R}{\pi E h} \sum_{n=1}^{\infty} q_n \cos kn\beta \int_0^\infty \frac{\cos \lambda \alpha}{\lambda^4 + c^2(kn)^8} \, d\lambda,
\]
from which, for the force factors, in accordance with (2), we have
\[
G_1^p = \nu G_2^p = \frac{\nu D}{\pi RE h} \sum_{n=1}^{\infty} q_n(kn)^6 \cos kn\beta \int_0^\infty \frac{\cos \lambda \alpha}{\lambda^4 + c^2(kn)^8} \, d\lambda, \tag{5}
\]
\[
T_1^p = -\frac{1}{\pi} \sum_{n=1}^{\infty} q_n(kn)^2 \cos kn\beta \int_0^\infty \frac{\lambda^2 \cos \lambda \alpha}{\lambda^4 + c^2(kn)^8} \, d\lambda.
\]
We restrict ourselves only to the force factors in the loaded cross section \( \alpha = 0 \), in which they reach maximum values. Taking into account that [7]
\[
\int_0^\infty \frac{\lambda^2 \, d\lambda}{\lambda^4 + c^2(kn)^8} = \frac{\pi \sqrt{2}}{4(kn)^2 \sqrt{c}}, \quad \int_0^\infty \frac{d\lambda}{\lambda^4 + c^2(kn)^8} = \frac{\pi \sqrt{2}}{4(kn)^6 c \sqrt{c}}, \tag{6}
\]
we obtain simpler expressions in place of (5):
\[
T_1^p = -\frac{1}{2 \sqrt{2}} (q(\beta) - q_0), \quad G_1^p = \nu G_2^p = \frac{\nu R \sqrt{c}}{2 \sqrt{2}} (q(\beta) - q_0). \tag{7}
\]
These relations are written with allowance for series (3).

Solutions of the boundary-effect equation that decrease at \( \pm \infty \) are well known [8] and will not be written here in complete form. In the loaded cross section \( \alpha = 0 \), they have the form
\[
T_1^b = 0, \quad T_2^b = -\frac{1}{2} q(\beta)^4 (1 - \nu^2) \sqrt{R/h}, \quad G_2^b = \nu G_1^b = \nu^4 q(\beta) \sqrt{R h} / \sqrt{3(1 - \nu^2)}. \tag{8}
\]