GENERALIZED MOMENT REPRESENTATIONS AND INVARIANCE PROPERTIES OF PADÉ APPROXIMANTS

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By the method of generalized moment representations, we generalize the well-known invariance properties of Padé approximants under linear-fractional transformations of approximated functions.

Let us first introduce necessary definitions and notation.

Definition 1 [1, p. 36]. Assume that a function \( f(z) \) can be expanded in a power series of the form

\[
f(z) = \sum_{k=0}^{\infty} s_k z^k
\]

in a neighborhood of the point \( z = 0 \).

Then the rational function

\[
\left[ \frac{M}{N} \right]_f(z) = \frac{P_M(z)}{Q_N(z)},
\]

where \( P_M(z) \) and \( Q_N(z) \) are algebraic polynomials of degrees \( \leq M \) and \( \leq N \), respectively, is called a Padé approximant of order \( [M/N] \) for the function \( f(z) \), provided that

\[
f(z) - \left[ \frac{M}{N} \right]_f(z) = O(z^{M+N+1}) \quad \text{as} \quad z \to 0.
\]

As a tool for the approximation of analytic functions, Padé approximants have significant advantages over polynomial approximations and are extensively used in numerical mathematics, number theory, and theoretical physics. There are many papers devoted to the investigation of the general properties of these approximants and, in particular, of their invariance properties under various transformations of approximated functions. The following result is presented in [1]:

Theorem 1 [1, p. 44]. Assume that, for an analytic function of the form (1), there exists a Padé approximant of order \( [M/N] \), \( N \in \mathbb{N} \cup \{0\} \). Then, for all \( a, b, c, \) and \( d \in \mathbb{R} \) such that \( c + ds_0 \neq 0 \), one can also construct a Padé approximant of the function \( \tilde{f}(z) = (a + bf(z))(c + df(z))^{-1} \) and, moreover, if \( \left[ \frac{N}{N} \right]_{\tilde{f}}(z) =: \tilde{P}_N(z)/\tilde{Q}_N(z) \), then \( \left[ \frac{N}{N} \right]_{\tilde{f}}(z) = \tilde{P}_N(z)/\tilde{Q}_N(z) \), where

\[
\tilde{P}_N(z) = aQ_N(z) + bP_N(z) \quad \text{and} \quad \tilde{Q}_N(z) = cQ_N(z) + dP_N(z).
\]

In 1981, Dzyadyk suggested a new approach to the investigation of Padé approximants based on the method of generalized moment representations [2].
**Definition 2** [2]. The generalized moment representation of a number sequence \( \{s_k\}_{k=0}^{\infty} \) in the Banach space \( X \) is defined as a collection of equalities

\[
s_{k+j} = \langle x_k, y_j \rangle, \quad k, j = 0, \ldots, \infty,
\]

where \( x_k \in X, \, k = 0, \ldots, \infty, \, y_j \in X^*, \, j = 0, \ldots, \infty, \) and \( \langle x, y \rangle \) denotes the value of a functional \( y \in X^* \) on an element \( x \in X \).

By applying and developing this approach, we managed to establish numerous properties and analyze the behavior of Padé approximants and their generalizations for many special functions [3–7]. It was also discovered that the application of generalized moment representations enables one to obtain relevant generalizations of the theorems on invariance of Padé approximants.

**Theorem 2.** Assume that, for an analytic function of the form (1), there exists a Padé approximant of order \([N-1/N], \, N \in \mathbb{N} \),

\[
[N-1/N]_f(z) = \frac{P_{N-1}(z)}{Q_N(z)}.
\]

Then, for the function

\[
\tilde{f}(z) = \left\{ f(z) \left[ z(1+\xi_{11}s_1) - s_0\xi_{11} \right] + s_0^2\xi_{11} \right\} \left\{ f(z) \left[ z^2(\xi_{01}\xi_{10}s_1 - \xi_{00} - \xi_{00}\xi_{11}s_1) - s_0 \right] + \left[ z(1+\xi_{10}s_0+\xi_{11}s_0 - \xi_{01} - \xi_{10}\xi_{01}s_0) - \xi_{11} \right] \right.
\]

\[
\left. + \left[ z(1+\xi_{10}s_0 + \xi_{11}s_0 - \xi_{01}\xi_{10}s_0 + \xi_{01}s_0 + \xi_{10}\xi_{01}s_0^2 + \xi_{01}s_0^2) + \xi_{11}s_0 \right] \right\}^{-1},
\]

the Padé approximant of order \([N-1/N] \) exists, provided that \( 1 + \xi_{10}s_0 + \xi_{11}s_1 \neq 0 \). Furthermore, the denominator \( \tilde{Q}_N(z) \) of this approximant can be represented in the form

\[
\tilde{Q}_N(z) = \frac{1}{1 + \xi_{10}s_0 + \xi_{11}s_1} \left\{ Q_N(z) \left[ 1 + (\xi_{10} + \xi_{01})s_0 + \xi_{11}s_1 + \right. \right.
\]

\[
\left. + \frac{1}{z} \xi_{11}s_0 - (\xi_{00}\xi_{11}s_0 - \xi_{01}\xi_{10}s_1)s_0 \right] + \left[ z(1+\xi_{10}s_0+\xi_{11}s_0 - \xi_{01} - \xi_{10}\xi_{01}s_0) - \xi_{11} \right]
\]

\[
\left. + \left[ z(1+\xi_{10}s_0 + \xi_{11}s_0 - \xi_{01}\xi_{10}s_0 + \xi_{01}s_0 + \xi_{10}\xi_{01}s_0^2 + \xi_{01}s_0^2) + \xi_{11}s_0 \right] \right\}^{-1},
\]

\[
\tilde{Q}_N(z) = 1 + \xi_{10}s_0 + \xi_{11}s_1 \cdot \sum_{j=0}^{N} c_j^{(N)}(\xi_{10}s_j + \xi_{11}s_{j+1}) + \sum_{j=0}^{N} c_j^{(N)}s_j,
\]

where \( c_j^{(N)}, \, j = 0, \ldots, N, \) are coefficients of the polynomial