Flow field due to a row of vortex and source lines spanning a conical annular duct

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Abstract. Analytical expressions are derived for the incompressible flow due to a row of vortex and source lines spanning a duct whose walls are coaxial conical surfaces of revolution with a common vertex. The singularity lines have the shape of an arc of circle meeting the walls perpendicularly, and are defined by the intersection of a spherical surface with a series of equally spaced meridional planes. Although source lines of spanwisely varying strength are in general assumed, only vortex lines of constant circulation are considered. Simpler expressions are derived for the limiting two-dimensional cases when the flow is axisymmetric (actuator disc), and when the angular distance between the conical walls becomes vanishingly small. The expressions for the latter case are used in an example to obtain numerical results by a panel method for the velocity distribution of the flow about the inlet guide vane system of a water turbine of bulb type.

1. Introduction

One of the approaches to the theoretical analysis of the internal flow in turbomachinery consists in representing the presence of blades (rotating and stationary) and struts by singularities of the vortex, source, sink and dipole types. Two of the earliest and best known methods using singularities to describe the two-dimensional plane flow about a rectilinear cascade of aerofoil profiles are due to Schlichting [1] and Martensen [2] (for a comprehensive review see [3] and [4]). The fully analytical solution of the three-dimensional problem has been attempted only for relatively simple and idealized geometries. In the case of a rectilinear cascade bounded by plane parallel walls, solutions have been obtained for the flow about twisted blades spanning the walls normally [5] or with sweep and dihedral effects [6]. The incompressible flow about an annular cascade of radial source and vortex lines of constant strength bounded by coaxial cylindrical walls was studied by Meyer [7] (see also [8]) who integrated Laplace's equation for the potential by separation of variables. This solution was extended by McCune [9] to source lines of radially varying strength in linearized compressible flow. In the case of lifting lines of spanwisely varying circulation, the problem is made more difficult by the presence of trailing vortices convected by the flow, as is well known in propeller theory (see e.g. [10]), and analytical solutions can in practice be obtained only if simplifying assumptions are introduced concerning the shape of the convected vortex filaments. In the case of a rotor in an annular cylindrical duct, the usual assumption consists in taking the free vortex lines to be of truly helical shape, building together helical vortex surfaces (small perturbation theory). Solutions for this problem were obtained by the present author [11] for incompressible flow and by Okurounmu and McCune [12] for linearized compressible flow.

Among the great variety of bladed ducts occurring in turbomachinery, it is not uncommon to find cases when the walls can be adequately modelled by conical surfaces, with the blades'
axes meeting the end walls at approximately right angles. As an example, we mention the inlet guide vane system of certain hydraulic turbines of bulb type (see e.g. [13]). In the present paper, theoretical results are derived for this kind of situation, with the lifting and thickness effects of the blades represented by vortex and source lines respectively. The flow is assumed incompressible, and irrotational outside the vortex singularities. In order to obtain exact analytical solutions, we further restrict the geometry of the inner and outer walls to conical surfaces of revolution with a common vertex, which, in a system of spherical co-ordinates \((r, \theta, \phi)\), are given by \(\theta = \text{constant}\). The vortex and source lines are circular arcs defined by \(r = \text{constant}\), \(\phi = \text{constant}\). The solution of Laplace’s equation for the velocity potential with the required boundary conditions is achieved by the method of separation of variables. Unlike in the case of screw propellers and other types of open turbomachines, constant blade circulation along the span is widely adopted as a design condition in closed turbomachines, which means a change in (circumferentially averaged) tangential velocity that is inversely proportional to the distance from the axis (constant blade work for a rotor). On the other side, lifting-lines of spanwisely varying circulation imply the presence of trailing vortex sheets, which in conical flow, even assuming small perturbation, are of considerably more complex shape than the helical sheets in cylindrical flow for which the solutions mentioned above were obtained. For these reasons, only lifting-lines of constant circulation are considered here. Such restriction does not apply to the source lines, which are assumed here in general to be of varying strength along the span.

In Section 3, we look for the form taken by the general expressions, derived in Section 2 for the three-dimensional case, when two limiting situations are reached in which the flow becomes two-dimensional. The first one (Section 3.1) is the axisymmetric flow resulting from taking the circumferential average of the velocity field, or equivalently the flow due to an actuator disc made up of vortex or source lines. The second case (Section 3.2) concerns a conical duct for which the angular distance between the walls becomes vanishingly small and the flow reduces to an infinitely thin layer whose thickness is proportional to the distance from the vertex. The expressions derived for the latter case are used, in Section 3.3, to obtain alternative expressions for the potential of the fully three-dimensional case, in the form of a superposition of an essentially two-dimensional potential and a singularity-free additional field representing the three-dimensionality.

Section 4 deals with the application of the analytical expressions to turbomachinery flow problems. In particular it is shown how they can be employed to obtain numerical results for the velocity field of the flow through the conical-walled inlet guide vane system of a water turbine.

2. Three-dimensional analysis

We consider the incompressible inviscid flow through a cascade of \(N\) equally spaced vortex lines or source lines spanning two conical walls, and choose a system of spherical co-ordinates \((r, \theta, \phi)\), with unit vectors \(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi\). The walls are at \(\theta = \theta_1, \theta = \theta_2\) \((0 < \theta_1 < \theta_2 < \pi)\). The singularity lines are defined by \(r = 1, \theta_1 \leq \theta \leq \theta_2, \phi = 2\pi n/N\) \((n = 0, 1, 2, \ldots, N - 1)\), their form being therefore an arc of circle (Fig. 1). The flow field is assumed irrotational everywhere except at the vortex lines and at the origin. We note that the radial flow, \(V_r = \text{constant}/r^2\), \(V_\theta = V_\phi = 0\), due to a source at the origin, and the swirling flow, \(V_\phi = \text{constant}/(r \sin \theta)\), \(V_r = V_\theta = 0\), due to a vortex along the axis of symmetry, both satisfy the