LOCAL ESTIMATES OF SOLUTIONS OF THE STATIONARY TWO-DIMENSIONAL FIRST BOUNDARY-VALUE PROBLEM OF MAGNETOHYDRODYNAMICS

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For solutions of a two-dimensional first boundary-value problem of magnetohydrodynamics, we obtain a priori asymptotic (for high Hartmann numbers) estimates of components of the velocity of a liquid and the stream function in the interior of the flow in spaces of continuous functions.

We consider a boundary-value problem which describes a stationary two-dimensional flow of viscous incompressible conductive liquid in a closed bounded multiply connected domain $D$. The boundary of the domain

$$S = \bigcup_{i=0}^{n} S_i$$

consists of finitely many closed contours ($S_0$ is the external contour) satisfying the Lyapunov condition; we assume that the boundary $S$ is impenetrable to liquid and ideally conductive. The medium outside $D$ has the magnetic permeability of a vacuum. The flow is generated by the motion of the liquid on $S$ with given velocity $v_0 \in C^2(S)$ directed along a tangent to $S$. The flow is subjected to the action of a constant potential magnetic field with induction $B$; we assume that the perturbation of the field $B$ by the flow is negligibly small (inductionless case).

1. Boundary-Value Problem

In the dimensionless and coordinateless form, the boundary-value problem has the form

$$(\nabla \times) ^2 v + \nabla (p + 0.5 R |v|^2) = -R \nabla \times (\nabla \times v) + H a^2 (v \times B) \times B,$$

$$\nabla \cdot v = 0, \quad v |_{S} = v_0, \quad v_0 \cdot v = 0,$$

where $v$ and $\tau$ denote the outer normal and the unit vector tangent to $S$, respectively, $\kappa = v \times \tau$, and $(\cdot)$ and $(\times)$ denote, respectively, scalar and vector products of vectors.

The vector induction field is represented in the form $B = B_d + \kappa b$, where the vector field $B_d \neq 0$ is coplanar to the plane of the flow. The field $B_d$ is assumed to be nonzero in $D \cup S$ and not tangent to $S$, i.e.,

$$0 < \inf_{D \cup S} |B_d| = m \leq |B_d| \leq \sup_{D \cup S} |B_d| < \infty,$$

$$\int_{S_i} B \cdot v dS_i = 0, \quad \int_{S_0} |B \cdot v|^2 dS_0 = 0.$$
takes constant values on the contours $S_j$ and is defined to within a constant, which can be fixed by using the condition $\psi|_{S_j} = 0$.

In [1], the following a priori estimates for solutions of problem (1) were proved:

$$
\| \psi \|_{L^2} \leq C_\psi(v_0, B_d, R, D, S) R H^{a-1/2},
$$
\[2\]

$$
\| \mathbf{v} \times B_d \|_{L^2} \leq C_B(v_0, B_d, D, S) R H^{a-1/2},
$$
\[3\]

$$
\| \mathbf{v} \|_{L^2} \leq C_v(v_0, B_d, R, D, S) R,
$$
\[4\]

$$
\| \nabla \times \mathbf{v} \|_{L^2} \leq C_{\nabla}(v_0, B_d, R, D, S) R H^{a-1/2}.
$$

An example constructed in [2] establishes the following estimate for the norm of $\psi$:

$$
\| \psi \|_{L^2} \leq M_\psi H^{-3/2}.
$$

This estimate, apparently, cannot be improved. In view of this fact, it is reasonable to consider the following estimate for $\psi$:

$$
\| \psi \| \leq M_\psi F_\psi(Ha), \quad H^{-3/2} \leq F_\psi(Ha) \leq Ha^{-1/2}.
$$

This inequality is the basis for establishing inner and local estimates of solutions of problem (1).

The main aim of this paper is to obtain asymptotic (for large values of the number $Ha$) estimates of the norms of the flow function $\psi$ and the field of velocities $\mathbf{v}$ in the space of continuous functions for problem (1). For this purpose, one can use any method for the investigation of the regularity of solutions of systems of nonlinear elliptic equations, in particular, the methods proposed by De Giorgi [3] and Moser [4] and their modifications [5]. The indicated estimates can be most simply obtained with the use of a local version of the Gehring lemma [6] and imbedding theorems in the form of multiplicative inequalities [5].

2. Multiplicative Inequalities

In [5], multiplicative inequalities are given for functions from the spaces $\tilde{W}^1_p(D)$. In the present paper, we use an analog of these inequalities for solenoidal vector fields. Let $\langle \mathbf{u}, \mathbf{v} \rangle$ denote the scalar product of vector fields in $L^2(D)$ and let $J(D)$ denote the set of finite solenoidal vector fields infinitely differentiable in $D$. Let $H(D)$ be the complement of $J(D)$ in the norm generated by the scalar product $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \nabla \times \mathbf{u}, \nabla \times \mathbf{v} \rangle$. Norms in the spaces $L_p$ are denoted by $\| \cdot \|_p$; below, for $p = 2$, we omit the index.

For the construction of a priori estimates of solutions of boundary-value problems of magnetohydrodynamics, a system of orthogonal curvilinear coordinates related to the field $B_d$ is usually used [1]. The Lamé coefficients for this coordinate system possess the property $m \leq \Lambda_1 = \Lambda_2 = |B_d| \leq m$.

**Theorem 1.** Let $q > 2$ and let a vector field $\mathbf{u}(u_1, u_2)$ belong to $H(D)$. Then, for $2 < q \leq 6$,

$$
\| u_1 \|_q \leq 4 m^{-1} \| u_1 \|^{1-\alpha} \| u_2 \|^{\alpha} \| \nabla \times \mathbf{u} \|^{2\alpha}, \quad \alpha = \frac{1}{2} - \frac{1}{q}.
$$