SOME GEOMETRIC-DIFFERENTIAL MODELS
IN THE CLASS OF FORMAL OPERATOR POWER SERIES

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We consider an example of a formal construction of local differential geometry in which smooth functions regarded as morphisms are replaced by formal operator power series.

1. Local relations in differential geometry are formulated in terms of smooth mappings of linear spaces (see [1], Chap. 2). The purpose of the present work is to show that these relations are not violated if, instead of the category of local manifolds, we use its modification in which smooth mappings regarded as morphisms are replaced by formal operator power series (see [2, 3]). This generalization may be useful for the solution of some problems in the theory of nonlinear partial differential equations, the investigation of which, as is known, involves methods of formal differential geometry (see [4, 5]).

2. Let us introduce basic objects and operations over them.

Let $X$ and $Y$ be linear topological spaces over a field $k$, let $L_k(X, Y)$ be a linear topological space of $k$-linear mappings from $X$ into $Y$, let $L_{k,s}(X, Y)$ be its subspace consisting of symmetric mappings, and let $L_{0,s}(X, Y) = Y$.

Consider the linear topological spaces

$$L(X, Y) = \prod_{k=1}^{\infty} L_k(X, Y) \quad \text{and} \quad L_s(X, Y) = \prod_{k=0}^{\infty} L_{k,s}(X, Y).$$

The elements of these spaces are infinite collections of the form

$$a = (a_1, a_2, \ldots, a_k, \ldots) \in L(X, Y), \quad a_k \in L_k(X, Y),$$

$$f = (f_0, f_1, \ldots, f_k, \ldots) \in L_s(X, Y), \quad f_k \in L_{k,s}(X, Y),$$

which are called formal mappings.

For simplicity, we denote the spaces $L(X, X)$ and $L_s(X, X)$ by $L(X)$ and $L_s(X)$, respectively. Denote by $\sim$ the projection of $L_s(X, Y)$ onto $L_s(X, Y) \cap L(X, Y)$, i.e.,

$$f = (f_0, f_1, \ldots, f_k, \ldots) \in L_s(X, Y) \Rightarrow \tilde{f} = (f_1, \ldots, f_k, \ldots).$$

Let us introduce the operation of composition

$$\circ : L(Y, Z) \times L(X, Y) \to L(X, Z),$$

$$(a \circ b)_k = \sum_{j=1}^{k} \sum_{k_1 + \ldots + k_j = k} a_j \circ \left( \bigotimes_{m=1}^{j} b_{k_m} \right).$$
where \(a_j\) is regarded as a linear mapping from \(Y^\otimes\) into \(Z\) (correspondingly, \(b_{km}\) is a linear mapping from \(X^\otimes_{km}\) into \(Y\)).

This operation is associative and linear with respect to the left component. Let \(I_X \in L_1(X, X) = L_1(X)\) denote the identity mapping of the space \(X\). The formal mapping \(\text{id}_X = (I_X, 0, \ldots, 0, \ldots)\) is the identity element with respect to the operation of composition. Thus, we obtain the category of formal mappings \(X\) whose objects are linear topological spaces and \(\text{Hom}(X, Y) = L(X, Y)\).

The elements of the set

\[
G(X, Y) = \{ a \in L(X, Y) \mid \exists a^{-1} \in L_1(Y, X) \}
\]

(and only these elements) are invertible, i.e., for \(a \in G(X, Y)\), there exists \(a^{-1} \in L(Y, X)\) such that \(a^{-1} \circ a = \text{id}_X\) and \(a \circ a^{-1} = \text{id}_Y\). Obviously, the set \(G(X) = G(X, X)\) is a group with respect to the operation of composition.

The element \(\varphi \in L(X, Y)\) generates a morphism of objects on \(X\) into objects on \(Y\) which is a homomorphism from \(L_s(Y, Z)\) into \(L_s(X, Z)\):

\[
\delta : L_s(Y, Z) \times L(X, Y) \to L_s(X, Z),
\]

\[
f \delta \varphi = f_0 \times S(f \circ \varphi),
\]

where \(S\) is the operation of symmetrization of a collection of multilinear mappings. It is easy to verify that \(f \delta (\varphi \circ \psi) = (f \delta \varphi) \circ \psi\).

For every element of the group \(G(X)\), we consider the morphism generated by its inverse element. Such morphisms are called changes of variables (or coordinate transformations). All changes of variables in \(L_s(X, Y)\) form a representation of the group \(G(X)\) in the linear topological space \(L_s(X, Y)\).

Assume that, for the elements of the spaces \(Y, Z, \) and \(V\), a bilinear operation of multiplication is defined, namely, \(Y \times Z \to V\). Let us extend it to the elements of the linear topological spaces \(L_s(X, Y)\) and \(L_s(X, Z)\):

\[
\cdot : L_s(X, Y) \times L_s(X, Z) \to L_s(X, V),
\]

\[
(f \cdot g)_k = S\left(\sum_{j+l=k} f_j \cdot g_l\right),
\]

where \(f \in L_s(X, Y), g \in L_s(X, Z),\) and

\[
(f_j \cdot g_l)(x_1, \ldots, x_k) = f_j(x_1, \ldots, x_j) \cdot g_l(x_{j+1}, \ldots, x_k).
\]

If \(\varphi \in L(X', X)\), then

\[
(f \cdot g) \delta \varphi = (f \delta \varphi) \cdot (g \delta \varphi).
\]

In particular, by using relation (1), we can extend the operation of multiplication of linear operators on the linear topological space \(E\) onto the space \(L_s(X, L_1(E))\). The elements of the subset