Abstract. This paper is devoted to the study of the Cauchy problem in $C^\infty$ and in the Gevrey classes for some second order degenerate hyperbolic equations with time dependent coefficients and lower order terms satisfying a suitable Levi condition.

1. Introduction

In this paper we shall consider the Cauchy problem

\begin{equation}
\begin{cases}
L(t, \partial_t, \partial_x)u(t, x) = 0, \\
u(0, x) = u_0(x), \\
\partial_t u(0, x) = u_1(x),
\end{cases}
\end{equation}

on $[0, T] \times \mathbb{R}^n_x$, where

\begin{align*}
L(t, \partial_t, \partial_x) &= \partial_t^2 - L_2(t, \partial_x) - L_1(t, \partial_x), \\
L_2(t, \partial_x) &= \sum_{i,j=1}^n a_{ij}(t) \partial_{x_i x_j}^2, \\
L_1(t, \partial_x) &= \sum_{j=1}^n b_j(t) \partial_{x_j},
\end{align*}

under the weak hyperbolicity condition

\begin{equation}
\sum_{i,j=1}^n a_{ij}(t)\xi_i \xi_j \geq 0 \quad \text{for all } (t, \xi) \in \mathbb{R} \times S^{n-1}.
\end{equation}
Let us define

\[ a(t, \xi) = \sum_{i,j=1}^{n} a_{ij}(t) \frac{\xi_i \xi_j}{|\xi|^2}, \]

\[ b(t, \xi) = \sum_{j=1}^{n} b_j(t) \frac{\xi_j}{|\xi|}. \]

We shall assume from now on that \( a_{ij} \in C^\infty(\mathbb{R}) \) and \( b_j \in C^0(\mathbb{R}) \). It is well known that the Cauchy problem (1) can fail to be \( C^\infty \)-well posed, even if \( b_j = 0 \), due to too fast oscillating coefficients (see [CS]); or, on the other hand, when the Levi condition is not satisfied by \( L_1 \), even if the coefficients \( a_{ij} \) are constants (see, e.g., [M]).

On the contrary, if \( L_2 \) is effectively hyperbolic, then (1) is \( C^\infty \)-well posed for any choice of \( L_1 \) (see [N2] and its bibliography). We observe that in this simple case the effective hyperbolicity of \( L_2 \) means that if for some \( (\bar{t}, \bar{\xi}) \in [0, T] \times S^{n-1} \) we have \( a(\bar{t}, \bar{\xi}) = 0 \), then

\[ \partial^2_\xi a(\bar{t}, \bar{\xi}) > 0. \]

The aim of this paper is to study the Cauchy problem (1) when the condition (5) is weakened to an assumption of finite degeneracy, and under a very precise Levi condition on the lower order term \( L_1 \). More precisely, we shall prove the following theorem.

**Theorem 1.** Assume that

\[ \sum_{j=0}^{\infty} |\partial^j_\xi a(t, \xi)| \neq 0 \quad \text{for all} \quad (t, \xi) \in [0, T] \times S^{n-1}. \]

Let \( k \) be the minimal integer satisfying

\[ \sum_{j=0}^{k} |\partial^j_\xi a(t, \xi)| \neq 0 \quad \text{for all} \quad (t, \xi) \in [0, T] \times S^{n-1}. \]

Suppose that there exist \( C > 0 \) and \( \gamma \in \left[0, \frac{1}{2}\right] \) such that

\[ |b(t, \xi)| \leq Ca(t, \xi)^\gamma \quad \text{for all} \quad (t, \xi) \in [0, T] \times S^{n-1}. \]

Then, if

\[ \gamma + 1/k < \frac{1}{2}, \]

the Cauchy problem (1) is well posed in \( \gamma^{(s)} \) for

\[ s \leq \frac{1 - \gamma}{\frac{1}{2} - (\gamma + 1/k)}. \]