Interpolating sequences in the ball of $\mathbb{C}^n$

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Abstract. Let $B$ be the unit ball of $\mathbb{C}^n$, I give necessary conditions on a sequence $S$ of points in $B$ to be $H^\infty(B)$ interpolating in term of a $\mathbb{C}^n$ valued holomorphic function zero on $S$ (a substitute for the interpolating Blaschke product).

These conditions are sufficient to prove that the sequence $S$ is interpolating for $\cap_{p>1} H^p(B)$ and is also interpolating for $H^p(B)$ for $1 \leq p < \infty$.

1. Introduction

Let $B$ be the unit ball of $\mathbb{C}^n$ and $S:=\{a_j\}_{j \in \mathbb{N}}$ be a sequence of points in $B$. We shall say that $S$ is $H\infty(B)$ interpolating if for every $\lambda:=\{\lambda_j\}_{j \in \mathbb{N}} \in \ell^\infty(\mathbb{N})$, there exists $f \in H^\infty(B)$ such that $f(a_j)=\lambda_j$ for all $j \in \mathbb{N}$.

We shall say that $S$ is $\cap_{p>1} H^p(B)$ interpolating if for every $\lambda:=\{\lambda_j\}_{j \in \mathbb{N}} \in \ell^\infty(\mathbb{N})$, there exist $f \in \cap_{p>1} H^p(B)$ such that $f(a_j)=\lambda_j$ for all $j \in \mathbb{N}$.

Finally we shall say that $S$ is $H^p(B)$ interpolating if for every $\lambda:=\{\lambda_j\}_{j \in \mathbb{N}}$ with $\|\lambda\|_p:=\sum_{j=0}^{\infty} |\lambda_j|^p(1-|a_j|^2)^n < +\infty$, there exists $f \in H^p(B)$ such that $f(a_j)=\lambda_j$ for all $j \in \mathbb{N}$.

If $S$ is $H^\infty(B)$ interpolating then the closed graph theorem gives the existence of a constant $C$ such that for any bounded sequence $\lambda$ there exists a function $f \in H^\infty(B)$ such that for all $j \in \mathbb{N}$, $f(a_j)=\lambda_j$ with the control $\|f\|_\infty \leq C\|\lambda\|_\infty$. The smallest such $C$ is called the interpolating constant of $S$.

The $H^\infty(B)$ interpolating sequences are precisely characterized for $n=1$ in the theorem of L. Carleson [8] and they are the same as the $H^p(B)$ interpolating sequences in that case [11]. Such a sequence is the set of zeros of an interpolating Blaschke product.

Let for $a \in \partial B$ and $h>0$, $Q:=Q(a,h):=\{\eta \in B ||1-\bar{a}\eta|<h\}$ be a pseudoball. We say that a measure $\mu$ on $B$ is a Carleson measure if there exist $C>0$ such that

$$|\mu|(Q(a,h)) \leq Ch^n$$

for all $a \in \partial B$ and $h > 0$. 

In the case \( n > 1 \), N. Varopoulos [13], proved that if \( S \) is interpolating for \( H^\infty (B) \) then the measure \( \mu = \sum_{j=1}^{\infty} \delta_{a_j} (1-|a_j|^2)^n \) is Carleson.

In [2], I proved that if \( S \) is \( H^2 \) interpolating, then again the measure \( \mu = \sum_{j=0}^{\infty} \delta_{a_j} (1-|a_j|^2)^n \) is Carleson and in [1], we proved that there is a sequence \( S \) in the ball of \( C^2 \) which is \( H^2 \) interpolating but not \( H^\infty \) interpolating, which means that the Varopoulos' condition is not sufficient for \( H^\infty \) interpolation.

On the other hand B. Berndtsson [7] proved that if the product of the Gleason distances of the points of \( S \) is bounded below, the sequence \( S \) is \( H^\infty \) interpolating. He also showed that this condition, which characterizes interpolating sequences when \( n = 1 \), is not necessary for \( n > 1 \).

The aim of this work is to give a generalization of the interpolating Blaschke product in the case of the ball in \( C^n \).

Let \( B \) be a \( C^n \) valued bounded holomorphic function in \( B \).

**Definition 1.1.** Let \( a \in B \), and \( \Phi_a \) be a biholomorphic map exchanging \( a \) and 0. We shall say that \( B \) is equivalent to \( \Phi_a \) near \( a \) if \( B = M_a \cdot \Phi_a \) with the matrix \( M_a \) invertible near \( a \). More precisely, we require that there is a \( \delta > 0 \) and \( C_B > 0 \) such that \( M_a \) is invertible in \( |\Phi_a| < \delta \) and, with \( A_a := M_a^{-1} \), \( |A_a| < C_B \) in \( |\Phi_a| < \delta \).

Now we can give the definition of an interpolating function for \( S \).

**Definition 1.2.** Let \( S := \{a_j\}_{j \in \mathbb{N}} \) be a sequence of points in \( B \) and \( B \) be a \( C^n \) valued bounded holomorphic function in \( B \). We say that \( B \) is interpolating for \( S \) if \( B \) is equivalent to \( \Phi_j := \Phi_{a_j} \) near \( a_j \) uniformly with respect to \( a_j \), i.e. the constants \( \delta \) and \( C_B \) are independent of \( a_j \).

Of course, if \( B \) is interpolating for \( S \) then it is zero on \( S \).

This is a characterization of the interpolating Blaschke products up to multiplication by a unit in \( H^\infty (D) \), if we add that \( S \) are the only zeros of \( B \).

The fact that this is a "possible" generalization in several variables is supported by the following theorems.

**Theorem 1.3.** Let \( B \) be the unit ball of \( C^n \), if the sequence \( S := \{a_j \in \mathbb{B}\}_{j \in \mathbb{N}} \) is interpolating for \( H^\infty (B) \) then there is an interpolating function \( B \) for \( S \).

**Theorem 1.4.** Let \( B \) be the unit ball of \( C^2 \), if there is an interpolating function \( B \) for the sequence \( S \), then the sequence \( S \) is \( \bigcap_{p > 1} H^p (B) \) interpolating.

**Theorem 1.5.** Let \( B \) be the unit ball of \( C^2 \), if there is an interpolating function \( B \) for the sequence \( S \), then the sequence \( S \) is \( H^p (B) \) interpolating for \( 1 \leq p < \infty \).

The sufficient results are stated and proved in \( C^2 \). No doubt they are true in \( C^n \), but at the price of non-trivial technical new results.

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