Vanishing residue characterization of the sine-Gordon hierarchy

François Treves

Abstract. The sine(hyperbolic)-Gordon hierarchy is shown to be the extension of the modified Korteweg–de Vries (MKdV) hierarchy in the integrodifferential algebra extending the standard differential algebra by means of one antiderivative. The characterization by vanishing residues of the MKdV hierarchy yields the same characterization of the sine(hyperbolic)-Gordon hierarchy in the integrodifferential algebra.

1. Introduction

This article is concerned with the hierarchy built upon the so-called sine-Gordon (SG) equation

\begin{equation}
\partial_t \partial_x u = \sin u.
\end{equation}

(1.1)

one of the classical soliton equations. Another soliton equation intimately related to (1.1) is the modified Korteweg–de Vries (MKdV) equation which plays an important role in the present work:

\begin{equation}
\partial_t u = \partial_x^2 u - 6u^2 \partial_x u.
\end{equation}

(1.2)

Each one of the equations (1.1) or (1.2) is the starting point of a sequence (called a hierarchy, see below) of evolution equations of increasing order $m$,

\begin{equation}
\partial_t u = P[ elf u].
\end{equation}

(1.3)

where

\[ u \mapsto P[u] = P(u, \partial u, \ldots, \partial^m u) \]

is a differential polynomial. Here $u$ is a smooth function of $t$ valued in a differential algebra $A$, i.e., a commutative algebra equipped with a derivation $\partial$ (also equipped
with a topology compatible with its differential algebra structure). In many applications \( A = C^\infty(S^1) \) (the periodic case) or \( A = \mathcal{S}(\mathbb{R}^1) \), the Schwartz space (the rapidly decaying case). Here, when we do make use of a differential algebra, it will be the algebra \( \mathbb{M}[[x]] \) of formal meromorphic series in one indeterminate \( x \).

\[
(1.4) \quad u(x) = \sum_{n=-\infty}^{\infty} a_n x^n.
\]

with coefficients in \( \mathbb{C} \) (\( N \in \mathbb{Z} \) may vary with \( u \), the derivation being the usual one, \( \partial_x = d/dx \). But mostly we shall reason at the symbolic level in the spirit of [GD1], the \( j \)th derivative \( \partial^j u \) being replaced by the symbol \( \xi_j \) and \( P(u, \partial u, ..., \partial^m u) \) by the true polynomial \( P(\xi) = P(\xi_0, \xi_1, ..., \xi_n) \).

We shall be primarily interested in the “conserved quantities” of the evolution equation (1.3), by which we mean most often other polynomials \( Q(\xi_0, ..., \xi_n) \) endowed with the following property:

1. \thinspace there is a polynomial \( \Phi(\xi_0, ..., \xi_r) \) (called a flux) such that

\[
(1.5) \quad \partial_t (Q[u]) = \partial (\Phi[u])
\]

for every solution \( u \in \mathcal{C}^1(\mathbb{R}; A) \) of (1.3).

The effect of (1.5) is that, when \( A = C^\infty(S^1) \) or \( A = \mathcal{S}(\mathbb{R}^1) \), then

\[
\int Q[u(t, x)] \, dx
\]

is a constant (“of motion”: the integration is carried out over \( S^1 \) or \( \mathbb{R}^1 \)).

The evolution equations (1.3) under consideration in this article have infinitely many (independent) conserved polynomials. The most famous of these equations is the Korteweg–de Vries (abbreviated KdV) equation. The MKdV and SG equations also admit infinite sequences of conserved polynomials and so do a number of other equations. Much attention, on the part of analysts, geometers and algebraists, has focused on those relatively few that have soliton solutions. On this vast subject we refer to the texts [AC], [D] and [FT]. There are equations, such as the Airy equation \( \partial_t u = \partial^3_x u \), which admit infinitely many conserved polynomials [for Airy these are the solution \( u \) and the “energies” \( \frac{1}{2} (\partial^k_x u)^2, k \in \mathbb{Z}_+ \)] but do not have soliton solutions.

Two differences between (1.1) and (1.3) jump to the eye: the right-hand side is a transcendental series, not a polynomial, and \( \partial_t u \) is replaced by \( \partial_t \partial_x u \). In the present work the first of those differences is dealt with by the routine extension from the algebra \( \mathfrak{B} = \mathbb{C}[\xi_0, \xi_1, ...] \) of polynomials in the (countable) infinity of indeterminates \( \xi_n \) to its natural completion \( \hat{\mathfrak{B}} \), the algebra of formal power series in the