THREE-DIMENSIONAL MIXED PROBLEMS OF THERMOELASTICITY FOR ISOTROPIC PLATES

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We obtain the homogeneous thermal solutions due to a temperature field for the three-dimensional thermoelastic problem for isotropic plates on whose plane faces homogeneous thermal and mixed mechanical conditions of flat face and diaphragm type are prescribed. This makes it possible to reduce the thermoelastic boundary-value problem to the corresponding elasticity problem. Bibliography: 6 titles

The author and others [1–3] have solved three-dimensional thermoelastic problems for isotropic plates on whose plane faces there are no stresses, displacements, or thermal actions. In the present article we study the thermostressed state of a plate on whose plane faces mixed boundary conditions are prescribed. The stressed state is due to thermal action on the lateral surface.

Consider an isotropic plate of thickness 2h whose boundary consists of two planes and the cylindrical surfaces of cavities that weaken it. The rectangular coordinate system \( \tilde{x}_1, \tilde{x}_2, \tilde{x}_3 \) is chosen so that the middle plane of the plate \( \tilde{x}_3 = 0 \) coincides with the \( \tilde{x}_1 \tilde{x}_2 \) coordinate plane. Along with the coordinates \( \tilde{x}_i \) (i = 1, 2, 3) we introduce dimensionless rectangular coordinates:

\[
x_1 = \tilde{x}_1/a, \quad x_2 = \tilde{x}_2/a, \quad x_3 = \tilde{x}_3/\lambda a, \quad \lambda = h/a,
\]

where \( a \) is the characteristic linear dimension of the plate in the plane.

Suppose the boundary conditions on the faces of the plate have the form

\[
u_j(x_1, x_2, \pm 1) = 0, \quad \sigma_{3j}(x_1, x_2, \pm 1) = 0 \quad (j = 1, 2) \tag{1}
\]

or

\[
u_3(x_1, x_2, \pm 1) = 0, \quad \sigma_{3j}(x_1, x_2, \pm 1) = 0, \quad \theta(x_1, x_2, \pm 1) = 0, \quad \theta'(x_1, x_2, \pm 1) = 0. \tag{2}
\]

Solving these thermoelastic problems reduces to integrating the equations of equilibrium and heat conduction [2]

\[
\lambda^{-2}u''_1 + D^2u_1 + \nu_1 \partial_1(\partial_1u_1 + \partial_2u_2) + \lambda^{-1} \nu_1 \partial_1u'_3 = (3\nu_1 - 1)\partial_1 \theta, \tag{3}
\]

\[
\lambda^{-2}u''_2 + D^2u_2 + \nu_1 \partial_2(\partial_1u_1 + \partial_2u_2) + \lambda^{-1} \nu_1 \partial_2u'_3 = (3\nu_1 - 1)\partial_2 \theta, \tag{4}
\]

\[
\lambda^{-1}(1 + \nu_1)u''_3 + \lambda D^2u_3 + \nu_1(\partial_1u'_1 + \partial_2u'_2) = (3\nu_1 - 1)\theta', \tag{5}
\]

\[
\lambda^2 D^2 \theta + \theta'' = 0 \tag{6}
\]

taking account of the boundary conditions on the lateral surface of the plate, the boundary conditions (1)–(4), and the equations of the Duhamel-Neumann law

\[
\sigma_{11} = \nu \nu_1 e + \partial_1u_1 - (1 + \nu)\nu_1 \theta, \quad 2\sigma_{12} = \partial_2u_1 + \partial_1u_2, \tag{7}
\]

\[
\sigma_{22} = \nu \nu_1 e + \partial_2u_2 - (1 + \nu)\nu_1 \theta, \quad 2\sigma_{13} = \partial_1u_3 + \lambda^{-1}u'_1, \tag{8}
\]

\[
\sigma_{33} = \nu \nu_1 e + \lambda^{-1}u'_3 - (1 + \nu)\nu_1 \theta, \quad 2\sigma_{23} = \partial_2u_3 + \lambda^{-1}u'_2. \tag{9}
\]

Here
\[
\begin{align*}
\partial_1 &= \partial/\partial x_1, \quad \partial_2 = \partial/\partial x_2, \quad D^2 = \partial^2_1 + \partial^2_2, \quad \nu_1 = (1 - 2\nu)^{-1}, \\
\varepsilon &= \partial_1 u_1 + \partial_2 u_2 + \lambda^{-1} u_3', \quad u_i = \ddot{u}_i/a, \quad \sigma_{ij} = \sigma_{ij}/(2G) \quad (i, j = 1, 2, 3), \quad \theta = \alpha T.
\end{align*}
\]
Also \(\ddot{u}_i, \sigma_{ij},\) and \(T\) are the dimensioned displacements, stresses, and temperature, \(G\) is the shear modulus, \(\nu\) and \(\alpha\) are respectively the Poisson ratio and the coefficient of linear thermal expansion, and the prime denotes the derivative with respect to \(x_3\).

Following Lur'e [4], we refer to the problem of a plate in a stressed state that is symmetric with respect to the middle plane as Problem A and the analogous problem with an asymmetric distribution of temperature and stress as Problem B.

Using the method discussed in [1-4], we obtain particular solutions of the equilibrium equations (5) (homogeneous thermal state) and the heat equation (6) taking account of the boundary conditions (1)-(4).

The homogeneous stressed state corresponding to boundary conditions (1) and (3) has the following form in Problem A:
\[
\begin{align*}
\theta(x_1, x_2, x_3) &= \sum_{k=1}^{\infty} Q_k(x_1, x_2) \cos \delta_k x_3, \\
u^j_1 &= (1 + \nu)\lambda^2 \sum_{k=1}^{\infty} \partial_j Q_k \delta_k^{-2} \cos \delta_k x_3 \quad (j = 1, 2), \\
u^3_1 &= (1 + \nu)\lambda^2 \sum_{k=1}^{\infty} Q_k \delta_k^{-1} \sin \delta_k x_3, \\
\sigma^{11}_1 &= -(1 + \nu)\lambda^2 \sum_{k=1}^{\infty} \partial^2_1 Q_k \delta_k^{-2} \cos \delta_k x_3, \\
\sigma^{12}_1 &= -(1 + \nu)\lambda^2 \sum_{k=1}^{\infty} \partial^2_2 Q_k \delta_k^{-2} \cos \delta_k x_3, \\
\sigma^{13}_1 &= -(1 + \nu)\lambda^2 \sum_{k=1}^{\infty} \partial_1 \partial_2 Q_k \delta_k^{-2} \cos \delta_k x_3, \quad \sigma^{13}_i \equiv 0 \quad (i = 1, 2, 3), \\
D^2 Q_k &= \delta_k^2 \lambda^{-2}, \quad \delta_k = \pi(2k - 1)/2.
\end{align*}
\]
For Problem B under the same conditions (1)-(3) the following equations hold:
\[
\begin{align*}
\theta(x_1, x_2, x_3) &= \sum_{k=1}^{\infty} Q_k(x_1, x_2) \delta_k^{-1} \sin \delta_k x_3, \\
u^j_1 &= (1 + \nu)\lambda^2 \sum_{k=1}^{\infty} \partial_j Q_k \delta_k^{-3} \sin \delta_k x_3 \quad (j = 1, 2), \\
u^3_1 &= -(1 + \nu)\lambda \sum_{k=1}^{\infty} Q_k \delta_k^{-2} \cos \delta_k x_3, \\
\sigma^{11}_1 &= -(1 + \nu)\lambda^2 \sum_{k=1}^{\infty} \partial^2_1 Q_k \delta_k^{-3} \sin \delta_k x_3, \\
\sigma^{12}_1 &= -(1 + \nu)\lambda^2 \sum_{k=1}^{\infty} \partial^2_2 Q_k \delta_k^{-3} \sin \delta_k x_3, \\
\sigma^{13}_1 &= -(1 + \nu)\lambda \sum_{k=1}^{\infty} \partial_1 \partial_2 Q_k \delta_k^{-3} \sin \delta_k x_3, \\
\sigma^{13}_i \equiv 0 \quad (i = 1, 2, 3), \quad D^2 Q_k = \delta_k^2 \lambda^{-2}, \quad \delta_k = k\pi.
\end{align*}
\]