THE STRESSED STATE OF A THREE-DIMENSIONAL PLATE WITH RIGID INCLUSIONS

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We study the stressed state of an isotropic plate with two rigid round inclusions stretched by uniform forces at infinity in the three-dimensional formulation. We give the results of numerical investigation. We exhibit the error of the corresponding applied theory. Four tables. Bibliography: 3 titles

Consider an isotropic plate of thickness 2h weakened by a series of curvilinear cavities with cylindrical surfaces \( G_i \) (\( i = 1, 2, \ldots \)) on which the displacements are prescribed. The plane faces of the plate are free of external forces.

We refer the plate to a dimensionless rectangular coordinate system \( O\xi\eta\zeta_1 \) such that the \( O\zeta_1 \)-plane coincides with the middle plane of the plate. Solving the problem of the stressed state of this plate reduces to integrating the system of equilibrium equations [1]

\[
\mu \Delta U + (\lambda + \mu) \nabla \cdot \nabla \mathbf{U} = 0
\]

(1)

under the following boundary conditions:

\[
\sigma_{G_i}|_{\zeta_1=\pm l} = \tau_{\xi\zeta_1}|_{\zeta_1=\pm l} = \tau_{\eta\zeta_1}|_{\zeta_1=\pm l} = 0,
\]

(2)

\[
U|_{G_i} = U_i(\xi, \eta, \zeta_1), \quad V|_{G_i} = V_i(\xi, \eta, \zeta_1), \quad W|_{G_i} = W_i(\xi, \eta, \zeta_1).
\]

(3)

Using A. I. Lur'e's symbolic method [2], one can represent the general stressed state of the body as the sum of three states: the potential, the turbulent, and the biharmonic. The displacements will then have the form

\[
2\mu U(\xi, \eta, \zeta_1) = -a \lambda \left\{ \frac{\partial \Psi^*}{\partial \xi} + x_1 \lambda^2 \left( \frac{1}{3} - \xi_1^2 \right) \frac{\partial \nabla^1 \Psi^*}{\partial \xi} + \right. \\
\left. \frac{1}{3\nu - 1} \frac{\partial \varphi^*}{\partial \xi} - 2\nu + \sum_{k=1}^{\infty} \cos \sigma_k \zeta_1 \frac{\partial B_k}{\partial \eta} - \sum_{p=1}^{\infty} n_p(\zeta_1) \frac{\partial C_p}{\partial \xi} \right\}.
\]

(4)

Here

\[
\lambda = \frac{h}{a}, \quad \nu_1 = \frac{1}{1 - 2\nu}, \quad x_1 = \frac{\nu}{2(\nu + 1)}, \quad \nabla^2 = \frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2},
\]

\( a \) is the characteristic linear dimension of the plate in the plane, \( \nu \) is the Poisson ratio, \( B_k(\xi, \eta) \) and \( C_p(\xi, \eta) \) are functions that satisfy the Helmholtz equation

\[
\nabla^2 \Phi - \frac{\alpha^2}{\lambda^2} \Phi = 0, \quad \alpha = \begin{cases} \sigma_k, & \Phi = B_k(\xi, \eta), \\ \gamma_p, & \Phi = C_p(\xi, \eta). \end{cases}
\]

(5)

\( \sigma_k = k\pi, \ 2\gamma_p \) are the roots of the equation \( z + \sin z = 0 \), and \( \varphi^*(\xi, \eta) \) is a harmonic function connected with the biharmonic function \( \Psi^*(\xi, \eta) \) by the relations

\[
\frac{\partial^2 \varphi^*}{\partial \xi^2} = -2\nu_1 \nabla^2 \Psi^*, \quad \frac{\partial^2 \varphi^*}{\partial \eta^2} = 2\nu_1 \nabla^2 \Psi^*.
\]

(6)

The solution (4) satisfies the system of equations (1) and the boundary conditions (2).

Expressing the biharmonic function \( \Psi^* (\xi, \eta) \) by Goursat's formula in terms of two analytic functions \( \varphi(z) \) and \( \chi(z) \) of a complex variable \( z = \xi + i \eta \), and taking account of relation (6), we obtain for the components of the displacement vector

\[
2 \mu (U + iV) = a \lambda \left\{ \chi \varphi(z) - (z - \bar{z}) \varphi'(z) + \chi(z) + 4 \chi_1 \lambda^2 \left( \zeta_1^2 - \frac{1}{3} \right) \varphi''(z) - \frac{2i \nu_1}{H} \sum_{k=1}^{\infty} \cos \sigma_k \zeta_1 \, dB_k + \sum_{p=1}^{\infty} \eta_p(\zeta_1) \, dC_p \right\},
\]

where \( \varphi = (3 - \nu)/(1 + \nu) \), and \( d \) is the Kolosov operator.

We now introduce a local orthogonal coordinate system \( n_1 \zeta_1 \) connected with one of the cylindrical surfaces \( G_i \). Here \( n \) is the coordinate along the normal to the curve \( \Gamma_i \), and \( s \) is arc length. Using the formulas for transition from one coordinate system to another \( [1] \), we obtain

\[
2 \mu U_n = a \lambda \left\{ M_{on} \Psi^* + \lambda^2 \left( \zeta_1^2 - \frac{1}{3} \right) M'_{on} \Psi^* + \frac{2i \nu_1}{H} \sum_{k=1}^{\infty} \cos \sigma_k \zeta_1 \frac{\partial B_k}{\partial s} + \sum_{p=1}^{\infty} \eta_p(\zeta_1) \frac{\partial C_p}{\partial n} \right\}.
\]

Here

\[
M_{on} \Psi^* = \text{Re} \sigma^{-1} \left[ \chi \varphi(z) - (z - \bar{z}) \varphi'(z) - \chi(z) \right],
\]

\[
M'_{on} \Psi^* = 4 \chi_1 \text{Re} \sigma \varphi''(z), \quad \sigma^* = e^{\alpha}, \quad H = 1 + nk(z),
\]

\( k(s) \) is the curvature of the curve \( \Gamma_i \), \( \alpha \) is the angle between the normal to \( \Gamma_i \) and the \( O\xi \)-axis, and \( B_k(n, s) \), \( C_p(n, s) \), \( \varphi(z) \), and \( \chi(z) \) are functions determined by solving Eq. (5) under the boundary conditions (3).

Using the asymptotic method of integration, we can represent the general integral of Eq. (5) as

\[
\Phi(n, s) = \sum_{k=0}^{\infty} \lambda^k \sum_{j=0}^{\infty} A_{kj}(s) \left( \frac{n}{H} \right)^j H^{-\frac{1}{2}} \exp \left( - \frac{n \alpha}{\lambda} \right),
\]

where \( A_{kj}(s) \) are coefficients defined recursively from the values \( [2] \)

\[
\Phi(0, s) = \varphi(s) = \sum_{k=0}^{\infty} \lambda^k \varphi_k(s).
\]

We shall assume that the boundary conditions (3) can be represented as

\[
U_n \bigg|_{n=0} = \lambda U_1(s, \zeta_1) + \lambda^2 U_2(s, \zeta_1) + \cdots.
\]

It can be shown that the following equalities hold:

\[
c_{p0}(s) = b_{k0}(s) = c_{p1}(s) = b_{k1}(s) = 0.
\]

where \( c_{p0}(s) \) and \( b_{k1}(s) \) are the boundary values for the functions \( C_p(n, s) \) and \( B_k(n, s) \) respectively.

Using formulas (9), one can easily derive from relations (8) the following asymptotic representations:

\[
2 \mu \frac{U_n}{a} \bigg|_{n=0} = \lambda M_{00} \Psi^*_1 + \lambda^2 \left[ M_{00} \Psi^*_1 - \sum_{p=1}^{\infty} \gamma_p \eta_p(\zeta_1) c_{p2}(s) \right] + \lambda^3 \left\{ M_{00} \Psi^*_2 - \sum_{p=1}^{\infty} \eta_p(\zeta_1) \left[ \gamma_p c_{p3}(s) + \frac{1}{2} k(s) c_{p2}(s) \right] + 2 \nu_1 \sum_{k=1}^{\infty} b'_{k2}(s) \cos \sigma_k \zeta_1 + \left( \zeta_1^2 - \frac{1}{3} \right) M'_{00} \Psi^*_0 \right\} + \cdots.
\]