A NUMERICAL-ANALYTIC METHOD OF STUDYING THE DISPERSION OF ELASTIC WAVES IN A FIXED ORTHOTROPIC CYLINDER WITH A SIMPLY CONNECTED CURVILINEAR SECTION

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We give a method of constructing the dispersion equations for waveguides made of rectilinearly orthotropic materials and having a section that is either elliptic or approximately that of a regular 2n-gon (n \geq 2) with rounded corners. The equations of motion for the waveguide can be integrated in series of basic particular solutions of exponential type. The dispersion functions are obtained in the form of reduced infinite determinants from homogeneous functional boundary conditions on the lateral surface of the waveguide after these conditions have been algebraized using orthogonal series expansions. Bibliography: 6 titles

A number of papers [1-5] have been devoted to theoretical studies of the spectra of normal elastic waves in cylindrical and prismatic waveguides of low-symmetry rectilinearly isotropic materials in the context of an exact three-dimensional model of dynamic strain. In most respects this problem remains open, especially in those cases in which there is no similarity between the directions of elastic symmetry of the material of the waveguide and the shape of a cross-section of it. In the present paper we propose a method of constructing the dispersion relations for orthotropic cylindrical waveguides having a simply connected section of elliptic shape (or a shape approximately that of a regular 2n-gon with rounded corners, n \geq 2) that is mirror-symmetric relative to the horizontal and vertical axes of a section.

Consider a waveguide occupying the region

\[ V = \{ (x_1, x_2) \in S, -\infty < x_3 < \infty \} \]

in rectangular coordinates \( O:z:ix_1x_3 \), where \( S \) is a simply connected section with boundary \( \Gamma \) defined by the parametric equation

\[ (x) \Gamma = \omega_n(\sigma) = R^*(\sigma + \epsilon_n \sigma^{-(2n+1)}), \quad \sigma = \exp(i\theta), \quad \theta \in [0, 2\pi]. \]

The boundary-value problem that describes the spectrum of normal waves in a cylinder with a fixed lateral surface (in a waveguide having ideal mechanical contact on the boundary with an absolutely rigid surrounding medium that resists it) includes the equations of motion [5] and the boundary conditions

\[ (u_1^*)_{\Gamma} = (u_2^*)_{\Gamma} = (u_3^*)_{\Gamma} = 0. \]

The notation used here and below is the same as in [5].

For the complex-valued elastic displacement functions in these waves we introduce the initial representation

\[ u_j^* = u_j(x_1, x_2)e^{-i(\omega t - kx_3)} \quad (j = 1, 3), \]

in which the complex amplitude functions \( u_j(x_1, x_2) \) can be represented in one of the following four forms:

\begin{align}
  u_1^{(1)} &= A_1^{(1)} \varphi_1^{(sc)}(x_1, x_2), & u_2^{(1)} &= A_2^{(1)} \varphi_1^{(ss)}(x_1, x_2), & u_3^{(1)} &= A_3^{(1)} \varphi_1^{(cc)}(x_1, x_2), \\
  u_1^{(2)} &= A_1^{(2)} \varphi_2^{(ss)}(x_1, x_2), & u_2^{(2)} &= A_2^{(2)} \varphi_2^{(sc)}(x_1, x_2), & u_3^{(2)} &= A_3^{(2)} \varphi_2^{(ss)}(x_1, x_2), \\
  u_1^{(3)} &= A_1^{(3)} \varphi_3^{(ss)}(x_1, x_2), & u_2^{(3)} &= A_2^{(3)} \varphi_3^{(sc)}(x_1, x_2), & u_3^{(3)} &= A_3^{(3)} \varphi_3^{(ss)}(x_1, x_2), \\
  u_1^{(4)} &= A_1^{(4)} \varphi_4^{(ss)}(x_1, x_2), & u_2^{(4)} &= A_2^{(4)} \varphi_4^{(sc)}(x_1, x_2), & u_3^{(4)} &= A_3^{(4)} \varphi_4^{(ss)}(x_1, x_2),
\end{align}

\[ \varphi^{(sc)}_j(x_1, x_2) = \sin \nu^{(j)}_1 x_1 \cos \nu^{(j)}_2 x_2, \]
\[ \varphi^{(ca)}_j(x_1, x_2) = \cos \nu^{(j)}_1 x_1 \sin \nu^{(j)}_2 x_2, \]
\[ \varphi^{(cc)}_j(x_1, x_2) = \cos \nu^{(j)}_1 x_1 \cos \nu^{(j)}_2 x_2, \]
\[ \varphi^{(sa)}_j(x_1, x_2) = \sin \nu^{(j)}_1 x_2 \sin \nu^{(j)}_2 x_2. \]

Here \( \nu^{(j)}_1, \nu^{(j)}_2, \) and \( A^{(j)}_p \) are arbitrary constants.

Substituting the representations (3) and (4)-(6) into the equations of motion for this waveguide leads to homogeneous systems of linear algebraic equations in the \( A^{(j)}_p \):

\[ \alpha^{(j)}_{m_1} A^{(j)}_{m_1} + \alpha^{(j)}_{m_2} A^{(j)}_{m_2} + \alpha^{(j)}_{m_3} A^{(j)}_{m_3} = 0 \quad (m = 1, 3), \]

where

\[ \alpha^{(j)}_{m_1} = \Omega^2 - c_{55} k^2 - c_{11} \left( \nu^{(j)}_1 \right)^2 - c_{66} \left( \nu^{(j)}_2 \right)^2, \]
\[ \alpha^{(j)}_{m_2} = \Omega^2 - c_{44} k^2 - c_{66} \left( \nu^{(j)}_1 \right)^2 - c_{22} \left( \nu^{(j)}_2 \right)^2, \]
\[ \alpha^{(j)}_{m_3} = \Omega^2 - c_{33} k^2 - c_{55} \left( \nu^{(j)}_1 \right)^2 - c_{44} \left( \nu^{(j)}_2 \right)^2, \]
\[ \alpha^{(j)}_{12} = i k - \eta_1 \left( c_{12} + c_{66} \nu^{(j)}_1 \nu^{(j)}_2 \right), \]
\[ \alpha^{(j)}_{13} = -\alpha^{(j)}_{31} = i k \eta_2 \left( c_{13} + c_{55} \nu^{(j)}_1 \nu^{(j)}_2 \right), \]
\[ \alpha^{(j)}_{23} = \eta_3 \nu^{(j)}_2. \]

\[ \eta_1 = \sin(\pi j - 5\pi/4)|\sin(\pi j - 5\pi/4)|^{-1}, \quad \eta_2 = (-1)^j, \quad \eta_3 = \sin(\pi j + \pi/4)|\sin(\pi j + \pi/4)|^{-1}. \]

The equalities

\[ \det \left| \alpha^{(j)}_m \right| = 0 \quad (10) \]

connect the parameters \( \nu^{(j)}_1 \) and \( \nu^{(j)}_2 \). If the values \( \left( \nu^{(j)}_1, \nu^{(j)}_2 \right) \) satisfy Eq. (10), then the following relations hold for \( A^{(j)}_p \):

\[ A^{(j)}_1 = \eta^{(j)}_1 A^{(j)}_3, \quad A^{(j)}_2 = \eta^{(j)}_2 A^{(j)}_3, \]
\[ \zeta^{(j)}_1 = \left( \alpha^{(j)}_{13} \alpha^{(j)}_{21} - \alpha^{(j)}_{11} \alpha^{(j)}_{23} \right) \Delta^{-1}_j, \]
\[ \zeta^{(j)}_2 = \left( \alpha^{(j)}_{23} \alpha^{(j)}_{12} - \alpha^{(j)}_{13} \alpha^{(j)}_{22} \right) \Delta^{-1}_j, \]
\[ \Delta_j = \alpha^{(j)}_{11} \alpha^{(j)}_{22} - \alpha^{(j)}_{12} \alpha^{(j)}_{21}. \]

Following the method of choosing and ordering the pair \( \left( \nu^{(j)}_1, \nu^{(j)}_2 \right) \) of [6], we introduce the countable sets \( \left\{ \left( \nu^{(j)}_1, \nu^{(j)}_2 \right) \right\}_{q=1}^{\infty} \) and the basic sets of amplitude functions corresponding to them, defined up to constant multiples \( A^{(j)}_{3q} \) by relations (4)-(7), (9) and (11).

Representing the functions \( u^*_q \) by series in the basic solutions of (4)-(7), we obtain from the boundary conditions (2) a set of homogeneous functional equations in the coefficients of these series. Orthogonal series can be effectively used to algebraize these functional equations. By combining the orthogonal expansions obtained in [6] for exponential functions on the curves \( \Gamma \) introduced by Eqs. (1).

\[ \left( \exp(\nu^{(j)}_1 x_1 + \nu^{(j)}_2 x_2) \right)_{l} = \sum_{(i)} Q_l(\nu^{(j)}_1, \nu^{(j)}_2) e^{i\theta}, \]
\[ Q_l(\nu^{(j)}_1, \nu^{(j)}_2) = \sum_{(s)} J_s \left( -i \varepsilon_n R \nu^{(j)}_1 x_1 \right) J_{l-2n} \left( -i \varepsilon_n R \nu^{(j)}_2 x_2 \right) \exp \left( i(l - (2n + 1)s) \delta_{1q} + \delta_{2q} \right). \]

\[ \exp(i\delta_{1q}) = \left( \nu^{(j)}_2 + i\nu^{(j)}_1 \right)^{-1}, \quad \exp(i\delta_{2q}) = \left( \nu^{(j)}_2 - i\nu^{(j)}_1 \right)^{-1}, \quad \nu^{(j)}_{12} = (\nu^{(j)}_1 + \nu^{(j)}_2)^{1/2}. \]