ACCURATE AND TRUNCATED THREE-POINT VARIATIONAL SCHEMES FOR THE STURM–LIOUVILLE PROBLEM WITH BOUNDARY CONDITIONS OF THE SECOND OR THIRD KIND

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On the basis of a variational approach for the eigenvalue problem in the case of a second-order linear differential operator with boundary conditions of the second or third kind, accurate and truncated three-point schemes of high order of accuracy are constructed. Error estimates of the approximate solution are obtained. The results of the numerical experiment are presented. Bibliography: 5 titles.

The solution of wide classes of problems of strained solid body mechanics requires searching for eigenvalues and eigenfunctions of both linear differential equations and their systems with boundary conditions of the second or third kind. Taking into account the fact that analytic solutions of such problems exist only in the case of constant coefficients, the construction, investigation, and usage of efficient computational methods of high order of accuracy for finding the eigenvalues and eigenfunctions of the Sturm–Liouville problem with arbitrary coefficients and with boundary conditions of the second or third kind are urgent.

In [1], the classic three-point accurate and truncated difference schemes of high order of accuracy for the eigenvalue problem with boundary conditions of the first kind that has the form

\[
\begin{align*}
(p(x)u')' - q(x)u + \lambda r(x)u &= 0, & x \in (0, 1), \\
u(0) &= u(1) = 0,
\end{align*}
\]

where the functions \( p(x), q(x), r(x) \in C^2(0, 1) \) satisfy the conditions

\[
0 < p_0 \leq p(x) \leq p_1, \quad 0 \leq q(x) \leq q_1, \quad 0 < r(x) \leq r_1, \quad p_0, p_1, q_1, r_0, r_1 \text{ — constants},
\]

are constructed and investigated.

In particular, it is shown in [1] that a truncated difference scheme of rank \( m \) with sufficiently small \( h \) has the \((2m + 2)\)th order of accuracy for eigenvalues and eigenfunctions in the norm of \( C[0, 1] \).

The results mentioned above are generalized in [2] to the case of an eigenvalue problem for vector systems of linear differential equations of the second order.

In [3], accurate and truncated variational schemes for the problem (1), (2) are constructed on the basis of the variational method, and it is proved that truncated variational schemes of the \( m \)th rank have the \((4m + 2)\)th order of accuracy in eigenvalues and the \((2m + 1)\)th order of accuracy in eigenfunctions in the norm of \( W_2^1[0, 1] \). It should be emphasized that the above-mentioned paper takes notice of the fact that the accurate variational scheme can be transformed into the classic accurate difference scheme of [1].

In this paper, the variational accurate and truncated schemes for the differential operator (1) and for boundary conditions of the second or third kind are constructed and investigated, and on their basis approximations to eigenvalues of the problem of \( SH \)-wave propagation in the sphere [5] are found.

Boundary conditions are considered in the form

\[
\begin{align*}
p(x)u'(0) &= -\alpha_1 u(0), & p(x)u'(1) &= \alpha_2 u(1), & \alpha_i \geq 0, & i = 1, 2.
\end{align*}
\]

Here, in the case \( \alpha_1 = \alpha_2 = 0 \), we have boundary conditions of the second kind, while in the case \( \alpha_1 > 0 \), \( \alpha_2 > 0 \) we have boundary conditions of the third kind.

Let us note that under the conditions (3) the Sturm–Liouville problem (1), (4) has an unlimited sequence of real eigenvalues $0 \leq \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \Rightarrow \infty$ to which there correspond a complete system of orthonormal eigenfunctions

$$ (u_k, u_m) = \int_0^1 r(x)u_k(x)u_m(x) \, dx = \delta_{km}. $$

For convenience, let us introduce the uniform grid

$$ \bar{\omega}_h = \{ x_i = ih, \ i = 0, \ldots, N, \ h = N^{-1} \} $$
on the interval $[0, 1]$.

**Definition.** The difference scheme of the form

$$ \Lambda^h y^h = a_iy^h_{i+1} + b_iy^h_i + c_iy^h_{i-1} = 0, \quad i = 1, \ldots, N - 1, $$

will be called an accurate three-point (classic or variational) scheme for the problem (1), (4) if the following conditions hold:

(a) $u(x_i, \lambda) = y^h_i$, $i = 0, \ldots, N$, where $\lambda$ is an eigenvalue and $u(x, \lambda)$ is the corresponding eigenfunction of the original problem (1), (4);

(b) coefficients of the operator $\Lambda^h$ are some local linear functionals of the coefficients of the differential operator (1).

On each of the intervals $[x_{i-1}, x_i]$, $i = 1, \ldots, N$, we consider the initial value problems

$$ (p(x)v'_k(x, \lambda))' - (q(x) - \lambda r(x))v_k(x, \lambda) = 0, \quad x \in [x_{i-1}, x_i], $$

$$ v_k(x_{i-1}, \lambda) = v_k^0(x_i, \lambda) = 0, \quad p(x)v'_k(x, \lambda) \big|_{x = x_{i-1}} = -p(x)v'_k(x, \lambda) \big|_{x = x_i} = 1, \quad i = 1, \ldots, N, \ k = 1, 2, $$

solutions of which $v_k^i(x, \lambda)$ will be called stencil functions. Since $v_k^i(x, \lambda)$ are solutions of Eq. (1), it is not difficult to show, using the properties of solutions of the problem of the form (1), (4), that for arbitrary $\lambda > 0$ and for sufficiently small $h$ the condition

$$ v^i_{i+p}(x_{i-p}, \lambda) \neq 0, \quad p = 0, 1, i = 1, \ldots, N, $$

is fulfilled. Hereafter we assume that (6) is always fulfilled.

Under these conditions the stencil functions $v^i_k(x, \lambda)$ are linearly independent solutions of Eq. (1).

The eigenfunction $u_k(x)$ of problem (1), (4) corresponding to the eigenvalue $\lambda_k$ can be written in the form

$$ u_k(x) = \frac{v'_k(x, \lambda_k)}{v'_i(x, \lambda_k)} u^k_i + \frac{v'_k(x, \lambda_k)}{v'_i(x_{i-1}, \lambda_k)} u^k_{i-1}, \quad x \in [x_{i-1}, x_i], $$

where $u^k_i = u_k(x_i)$, $i = 1, \ldots, N$.

It is possible to show that problem (1), (4) is equivalent to the variational problem: it is necessary to find a pair $(\mu, v(x))$, $\mu \in R$, $v(x) \in W^1_2(0, 1)$ such that the equality

$$ I(u, v) = \int_0^1 [p(x)u'v' - (q(x) - \mu r(x))uv] \, dx + \alpha_1 u(0)v(0) + \alpha_2 u(1)v(1) = 0 $$

is fulfilled for an arbitrary function $u(x)$ belonging to the space $W^1_2(0, 1)$.

Let us introduce the space of functions $B_k(0, 1)$, $k < N$:

$$ W(x) \in B_k(0, 1) \iff \exists g_i \in R, \quad i = 0, \ldots, N, $$

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